London–Oxford–Paris TDA Seminar – 06/11/2025

Simplicial approximation, Delaunay triangulations, and the list homomorphism problem

Raphaël Tinarrage – IST Austria

Consider data X and an auxiliary space Y.

$[X, Y]$ = homotopy classes of maps from X to Y.

Y	**Property**	**Applications**
S^1	$[X, S^1] \simeq H^1(X, \mathbb{Z})$	Circular coordinates (de Silva, Morozov, Vejdemo-Johansson, 2011) (Perea, 2020)
Real projective space $\mathbb{R}P^\infty$	$[X, \mathbb{R}P^\infty] \simeq H^1(X, \mathbb{Z}/2\mathbb{Z})$	Projective coordinates, `DREiMac` (Perea, 2018) (Polanco, Perea, 2019) (Perea, Scoccola, Tralie, 2023)
Lens space S^∞/p	$[X, S^\infty/p] \simeq H^1(X, \mathbb{Z}/p\mathbb{Z})$	
Complex projective space $\mathbb{C}P^\infty$	$[X, \mathbb{C}P^\infty] \simeq H^2(X, \mathbb{Z})$	
Grassmannian $\mathcal{G}(d, \mathbb{R}^\infty)$	$[X, \mathcal{G}(d, \mathbb{R}^\infty)] \simeq$ rank-d vector bundles on X	Persistent characteristic classes (T, 2022) (Scoccola, Perea, 2023) (Gang, 2025)

3-manifolds: `Regina`, `SnapPy`, `Twister`, ...

Space	**Known triangulations**	**References**
Real projective space $\mathbb{R}P^d$	$d \geq 1$	(Kühnel, 1987) (Adiprasito, Avvakumov, Karasev, 2022)
Complex projective space $\mathbb{C}P^d$	$d \geq 1$	(Sergeraert, 2010) (Sarkar, 2014) (Datta, Spreer, 2024)
Special orthogonal group $\mathrm{SO}(d)$	$d \leq 4$	$\mathrm{SO}(3) \simeq \mathbb{R}P^3, \quad \mathrm{SO}(4) \simeq S^3 \times \mathrm{SO}(3)$
Special unitary group $\mathrm{SU}(d)$	$d \leq 2$	$\mathrm{SU}(2) \simeq S^3$
Unitary group $\mathrm{U}(d)$	$d = 1$	$\mathrm{U}(1) \simeq S^1$
Stiefel manifold $\mathcal{V}(d, \mathbb{R}^n)$	$d = 1$ or $n \leq 4$	$\mathcal{V}(1, \mathbb{R}^n) \simeq S^{n-1}, \quad \mathcal{V}(d, \mathbb{R}^d) \simeq \mathrm{O}(d)$
Grassmannian $\mathcal{G}(d, \mathbb{R}^n)$	$d = 1$ or $n-1$	$\mathcal{G}(1, \mathbb{R}^n) \simeq \mathcal{G}(n-1, \mathbb{R}^n) \simeq \mathbb{R}P^{n-1}$

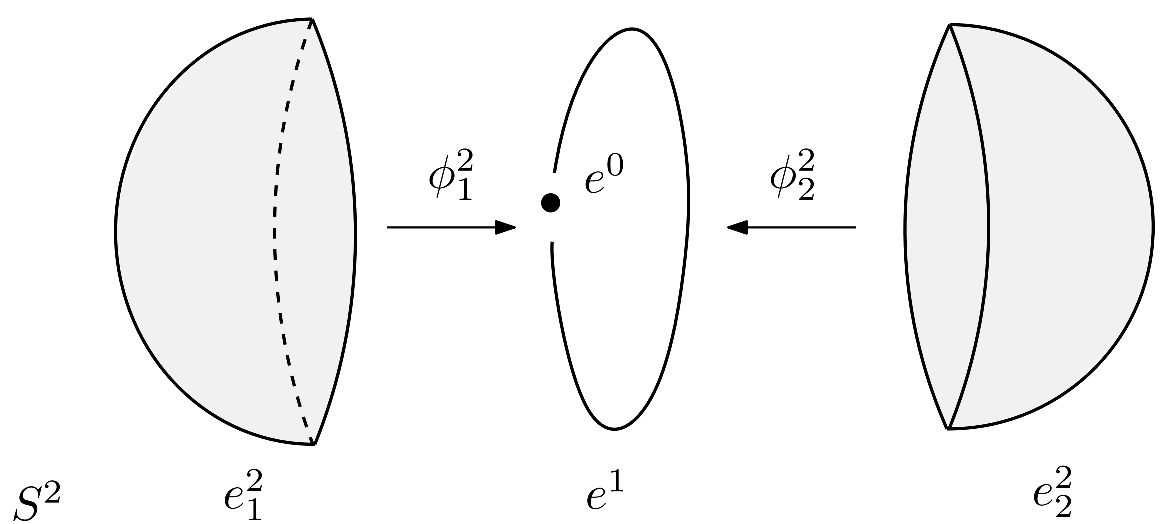
CW complex: Space X with a decomposition $X = X^d \supset X^{d-1} \supset X^{d-2} \supset \cdots$ such that

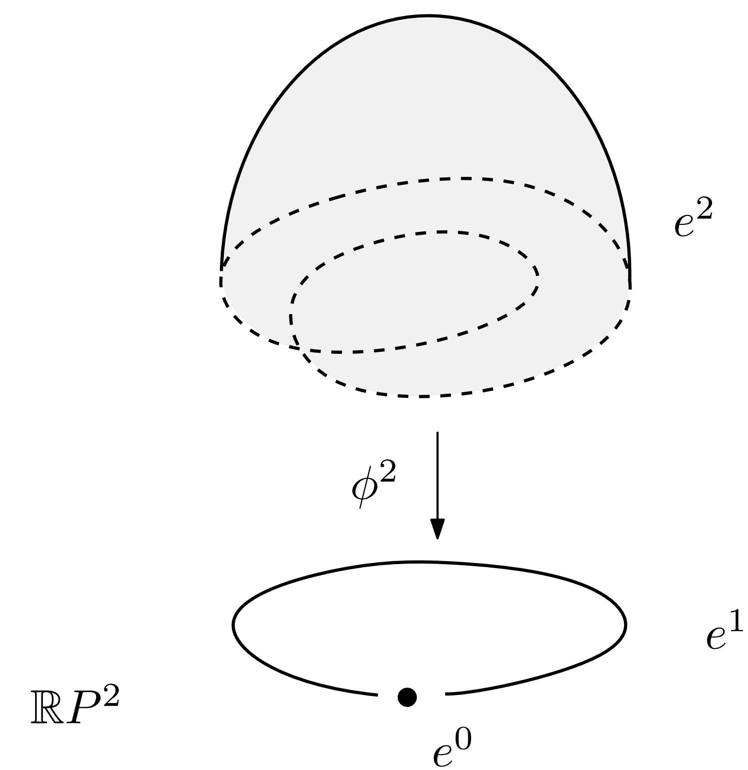
$$X^k = X^{k-1} \coprod_{1 \le i \le n(k)} e_i^k \qquad \text{where} \qquad e_i^k \simeq \mathring{B}^k \text{ (open ball).}$$

Characteristic map: $\Phi_i^k \colon B^k \to \overline{e}_i^k$

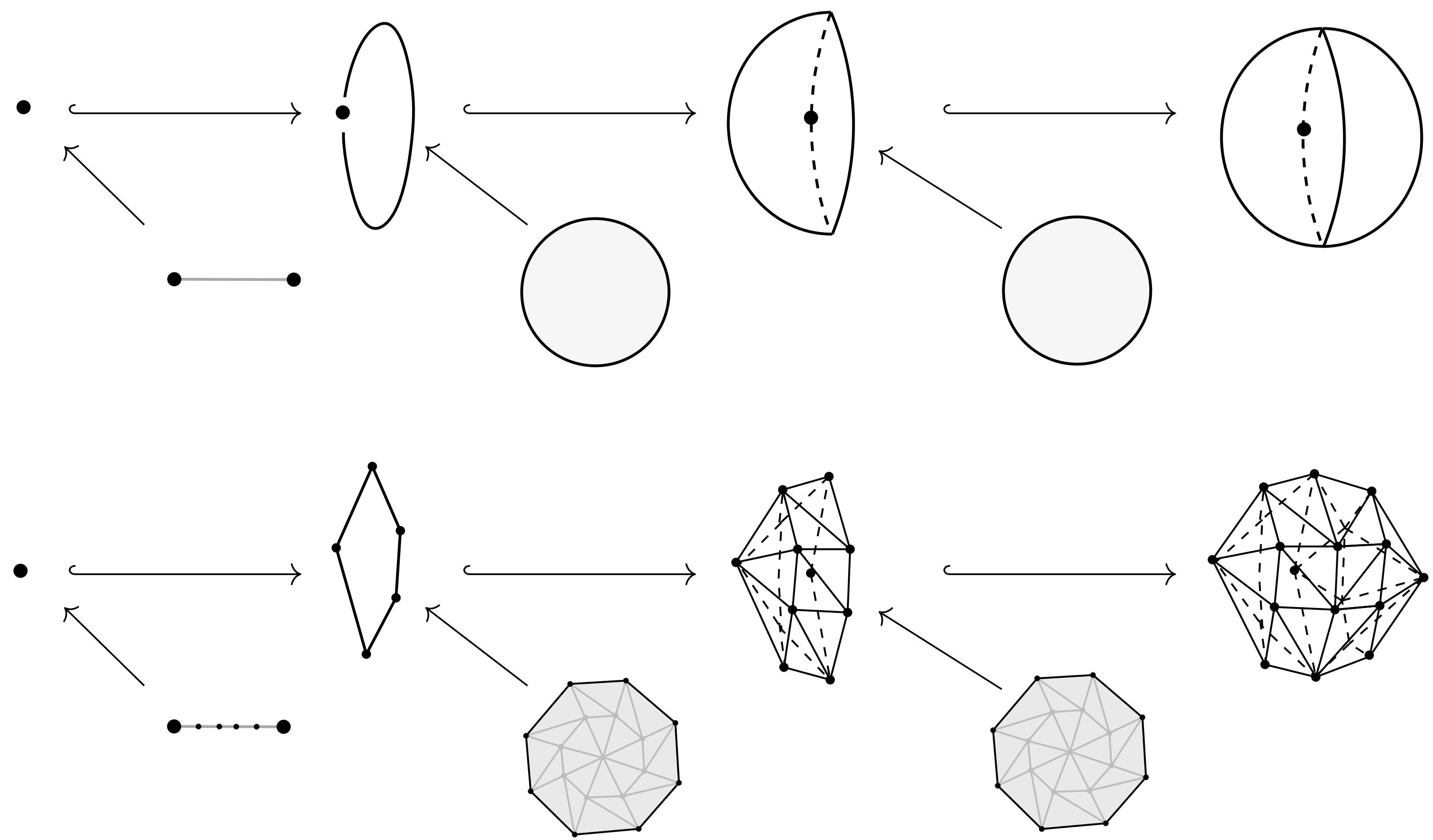
Gluing map: $\phi_i^k \colon S^{k-1} \to X^{k-1}$

In other words, $X^k = X^{k-1} \cup_{\phi_1^k} B^k \cup_{\phi_2^k} B^k \cup \cdots$



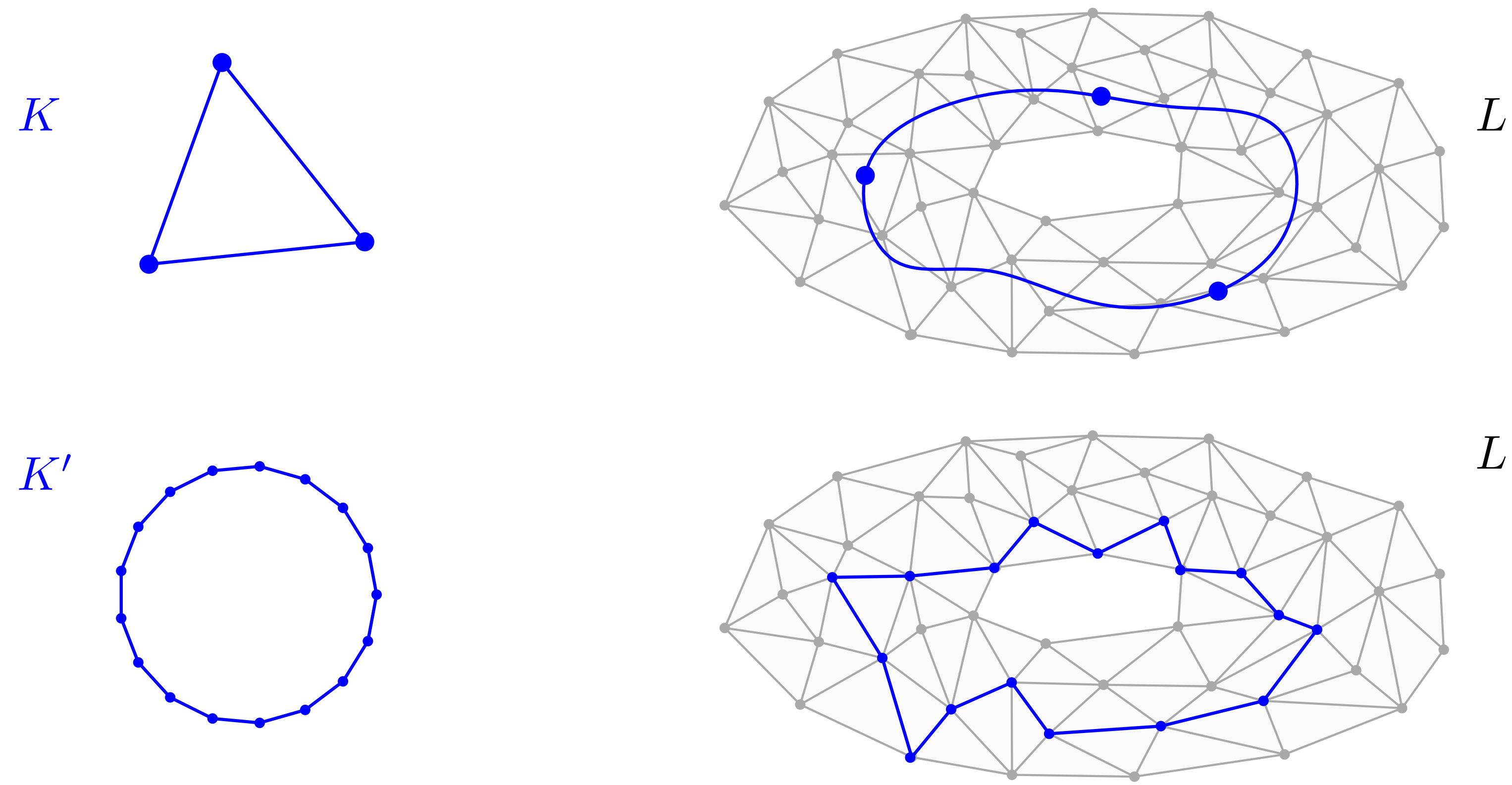


$$\begin{tikzcd}
e^0 \arrow[rr, hook] & & e^0 \cup e^1 \arrow[rr, hook] & & e^0 \cup e^1 \cup e_1^2 \arrow[rr, hook] & & e^0 \cup e^1 \cup e_1^2 \cup e_2^2 \\
& \partial B^1 \arrow[ul, "\phi^1"] & & \partial B^2 \arrow[ul, "\phi_1^2"] & & \partial B^2 \arrow[ul, "\phi_2^2"] &
\end{tikzcd}$$



Consider simplicial complexes K, L and a continuous map $f\colon |K| \to |L|$ between geometric realizations.

Is there a simplicial map $g\colon K \to L$ homotopic to f?



Simplicial approximation

Consider simplicial complexes K, L and a continuous map $f\colon |K| \to |L|$ between geometric realizations.

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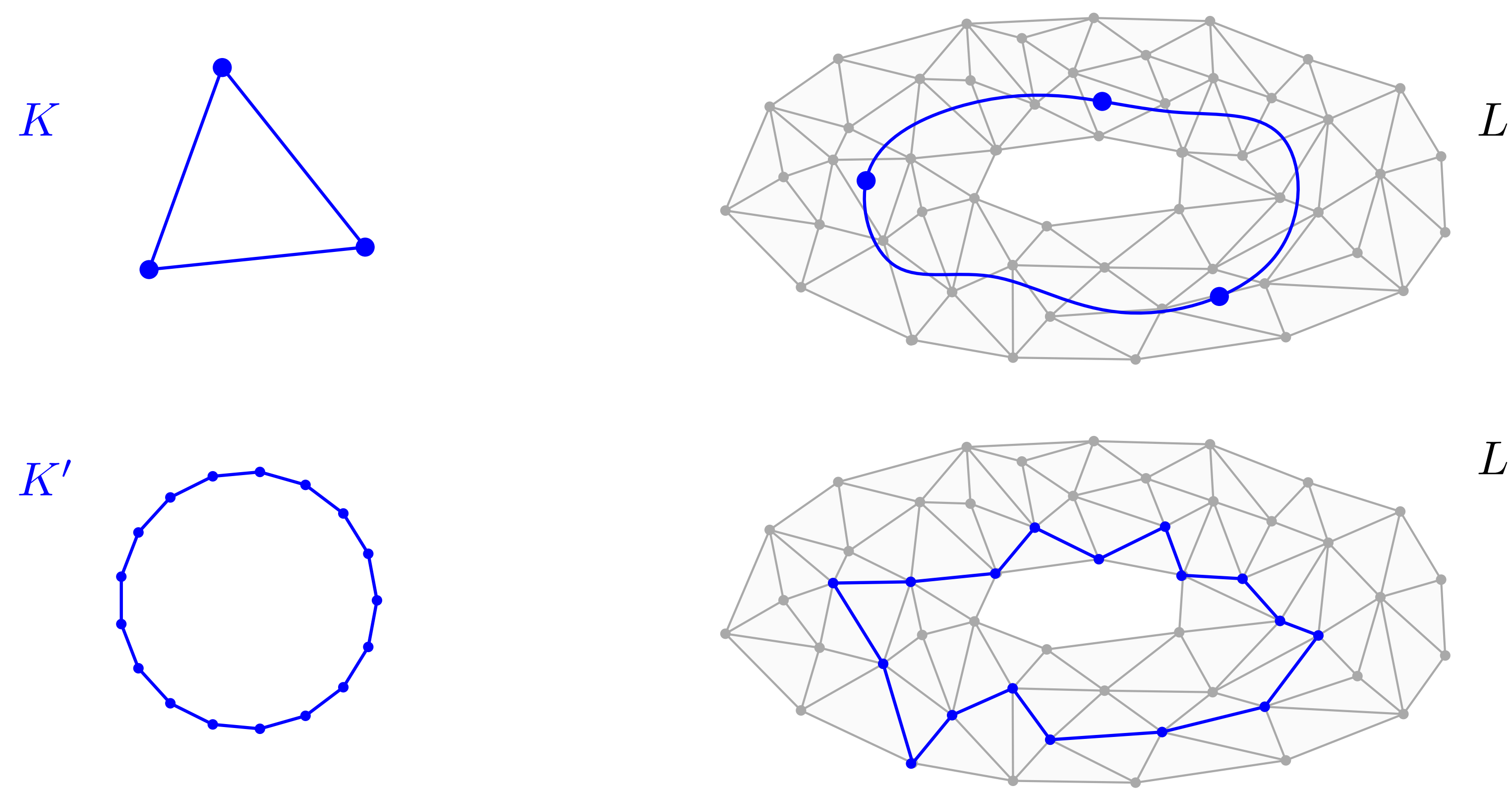
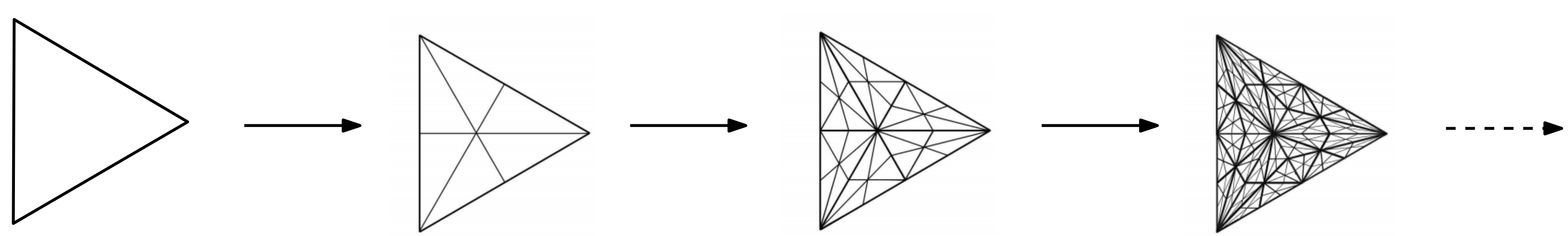


Simplicial Approximation Theorem: Yes, after a certain number of barycentric subdivisions on K.

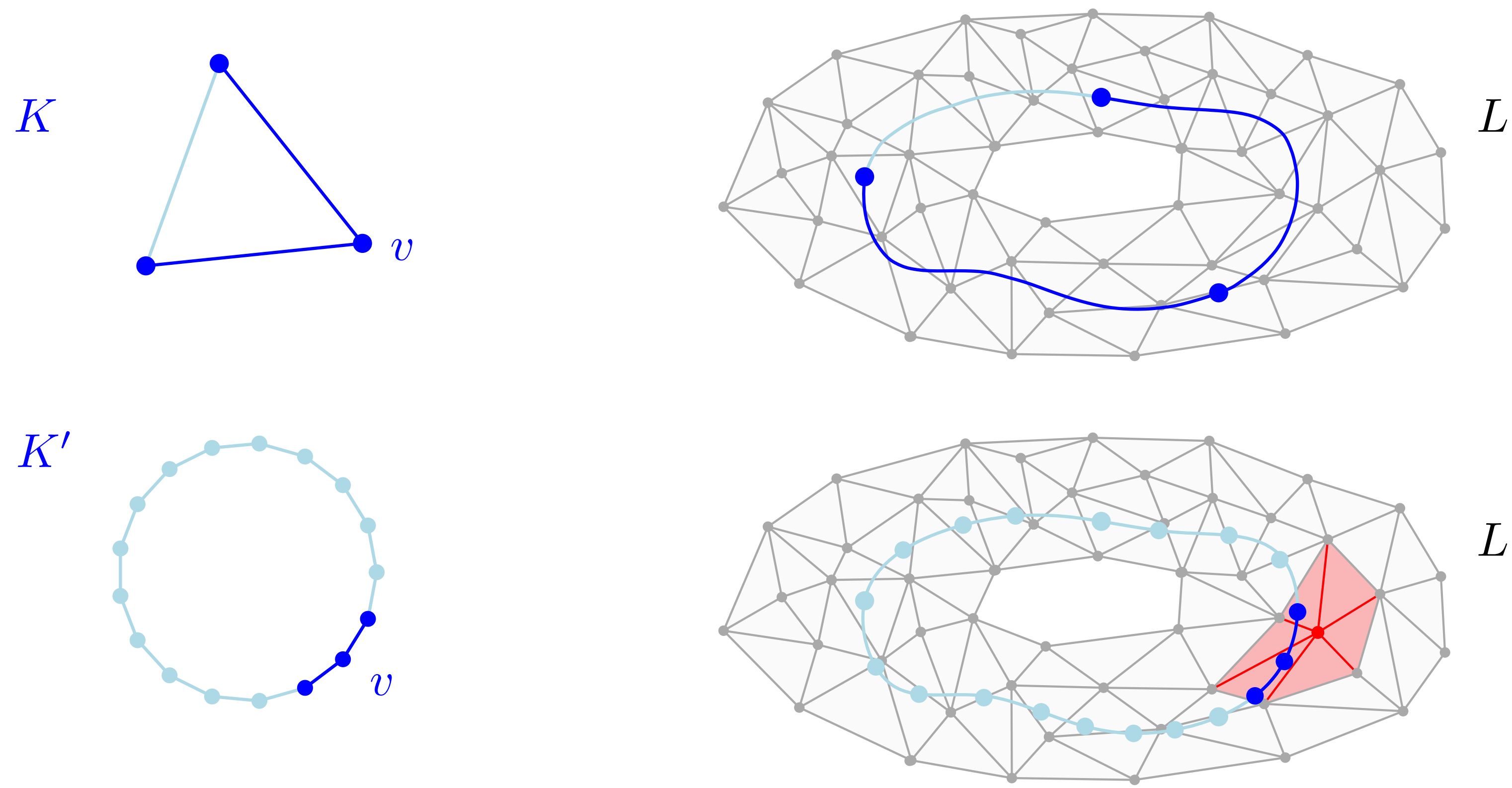


Star condition

Given a vertex $v \in V(K)$, consider the open star: $\mathrm{St}\,(v) = \{\sigma \in K \mid v \in \sigma\}$,

closed star: $\overline{\mathrm{St}}\,(v) = \{\tau \in K \mid \exists \sigma \in \mathrm{St}\,(v)\,,\ \tau \subset \sigma\}$.

The map $f\colon |K| \to |L|$ satisfies the **star condition** if $\forall v \in V(K)$, $\exists w \in V(L)$ s.t. $f\left(\left|\overline{\mathrm{St}}\,(v)\right|\right) \subseteq |\mathrm{St}\,(w)|$.



Every assignement $v \mapsto w$ defines a simplicial map g homotopic to f.

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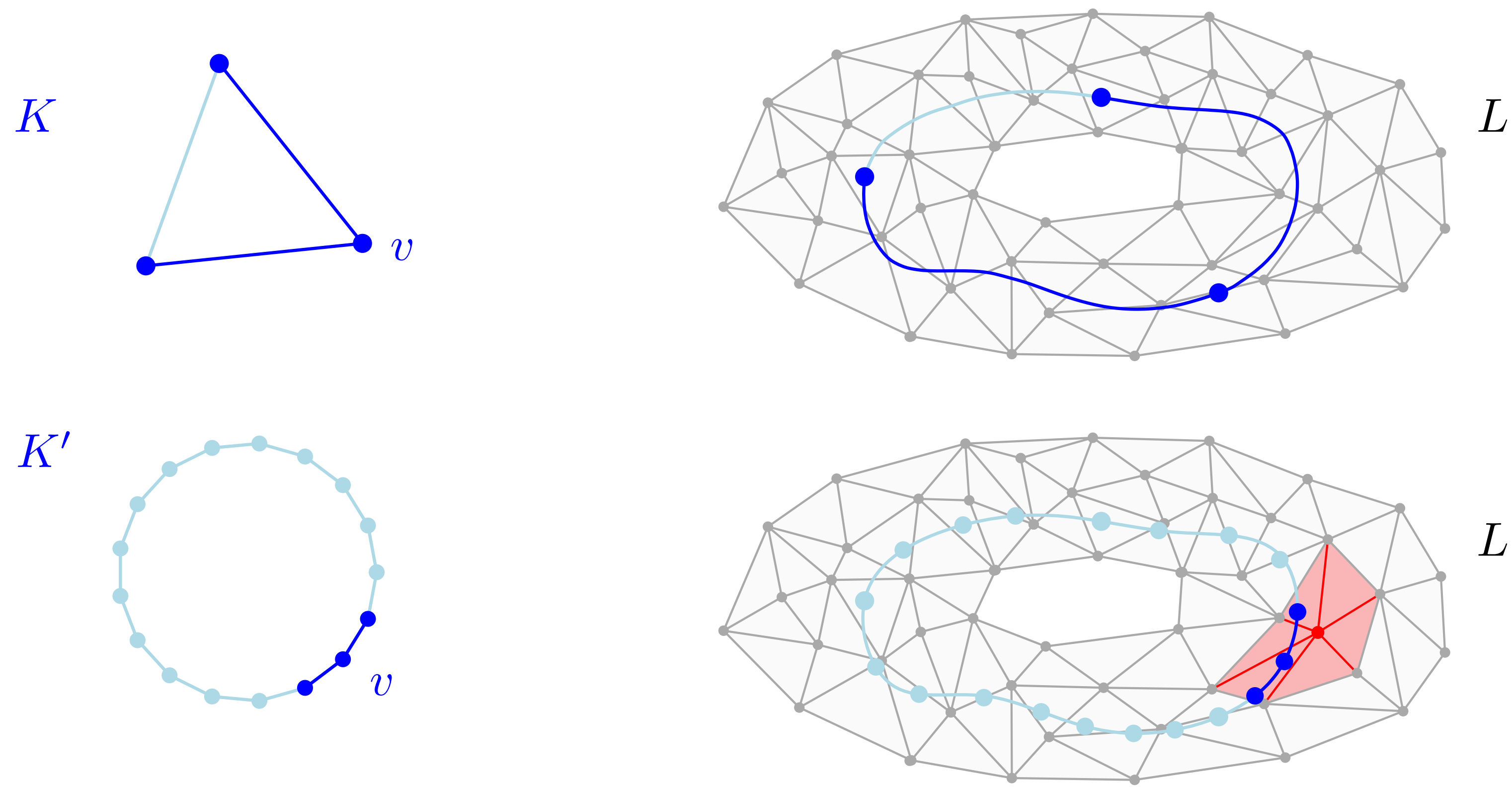
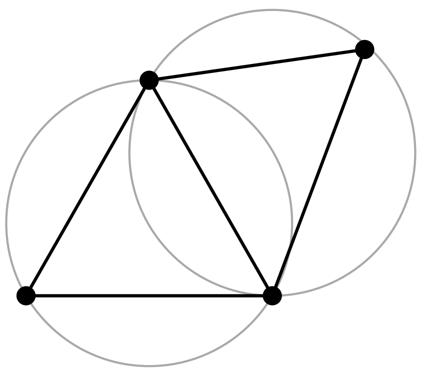


Every assignement $v \mapsto w$ defines a simplicial map g homotopic to f.

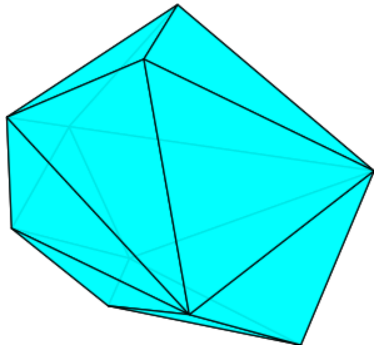
Problem: Barycentric subdivision turns a d-simplex into a complex with $2^{d+1}-1$ vertices and $(d+1)!$ simplices.

Let $X \subset S^d$ finite.

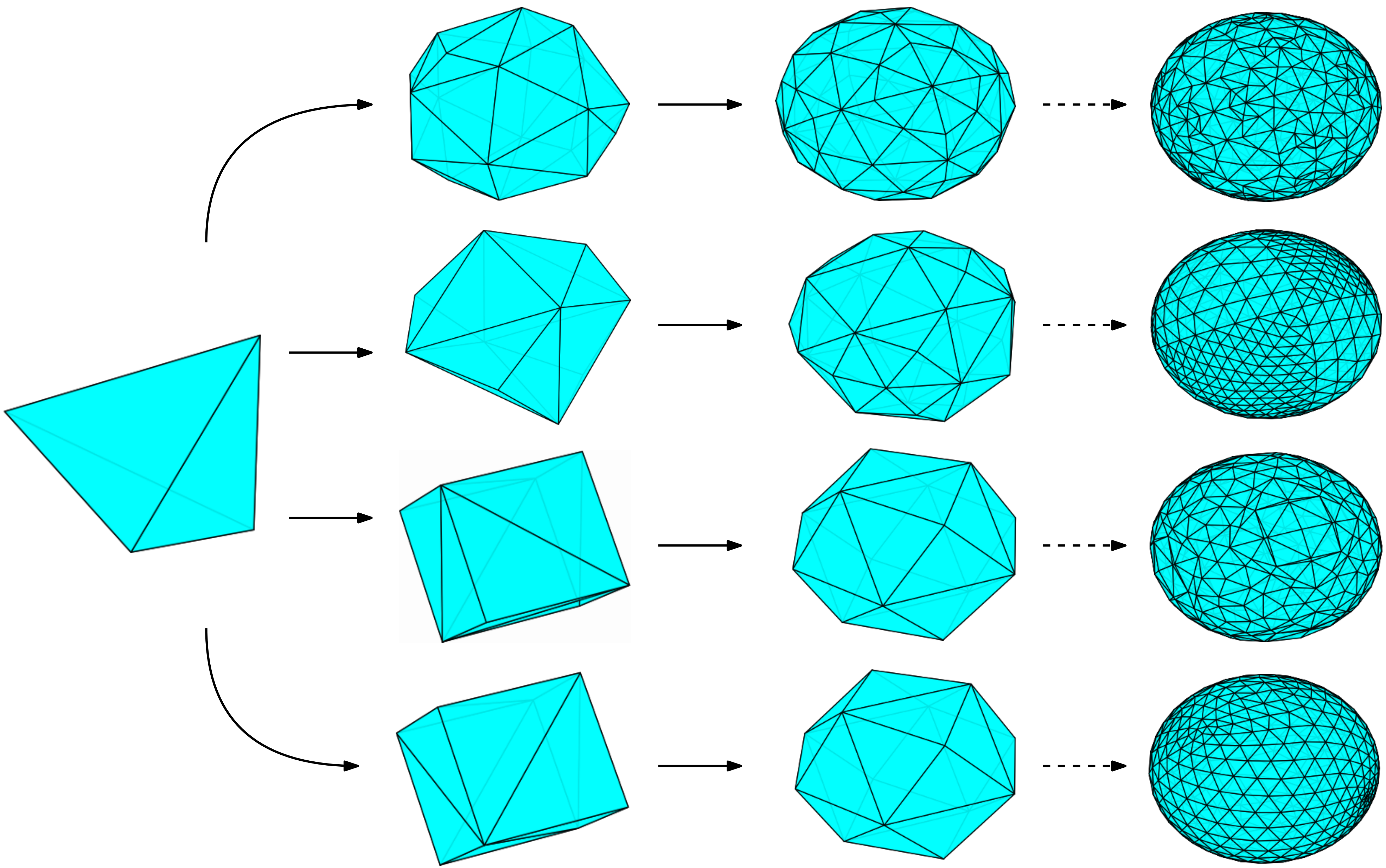
Intrinsic definition: The facets of $\mathrm{Del}(X)$ are the subsets of $d+1$ points whose circumscribing open ball is empty of points of X.



Extrinsic definition: $\mathrm{Del}(X)$ coincides with the boundary of the convex hull of X.



- In $\mathbb{R}^2$, $\mathrm{Del}(X)$ maximizes the minimum angle over all triangulations of X (Sibson, 1978);
- In $\mathbb{R}^n$, $\mathrm{Del}(X)$ minimizes the maximal miniradius of the simplices (Rajan, 1991);
- In $\mathbb{R}^n$, $\mathrm{Del}(X)$ minimizes a certain weighted sum of edge lengths (Musin, 1997).



Steiner points

barycenter of all simplices

midpoint of edges

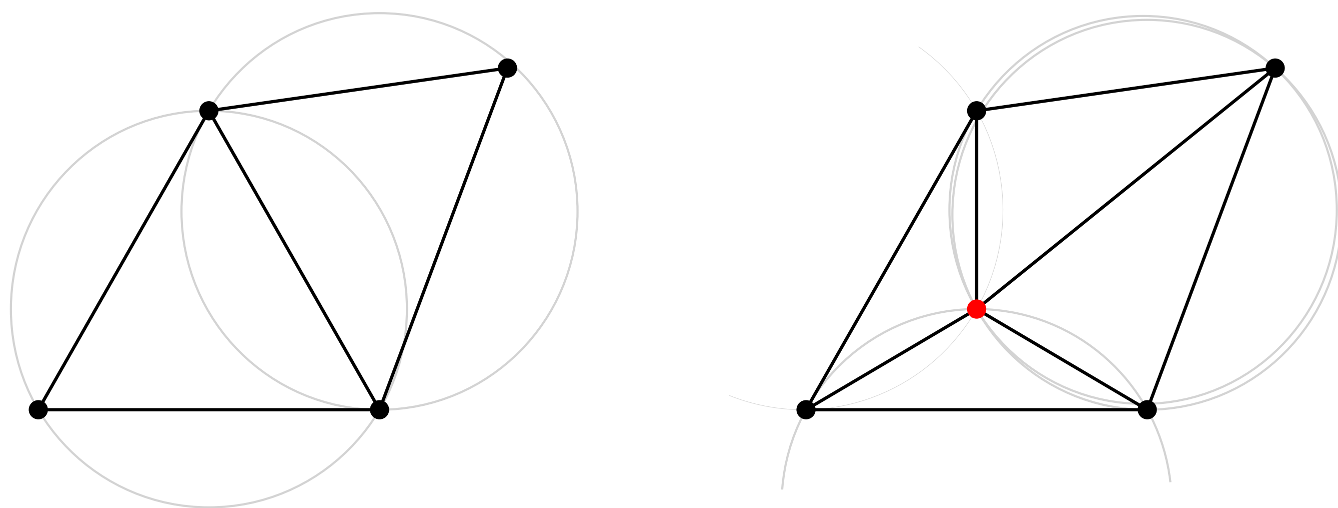
minicenter of facets

centroid of facets

Delaunay refinements

To prove: the **maximal diameter** of simplices tends to zero.

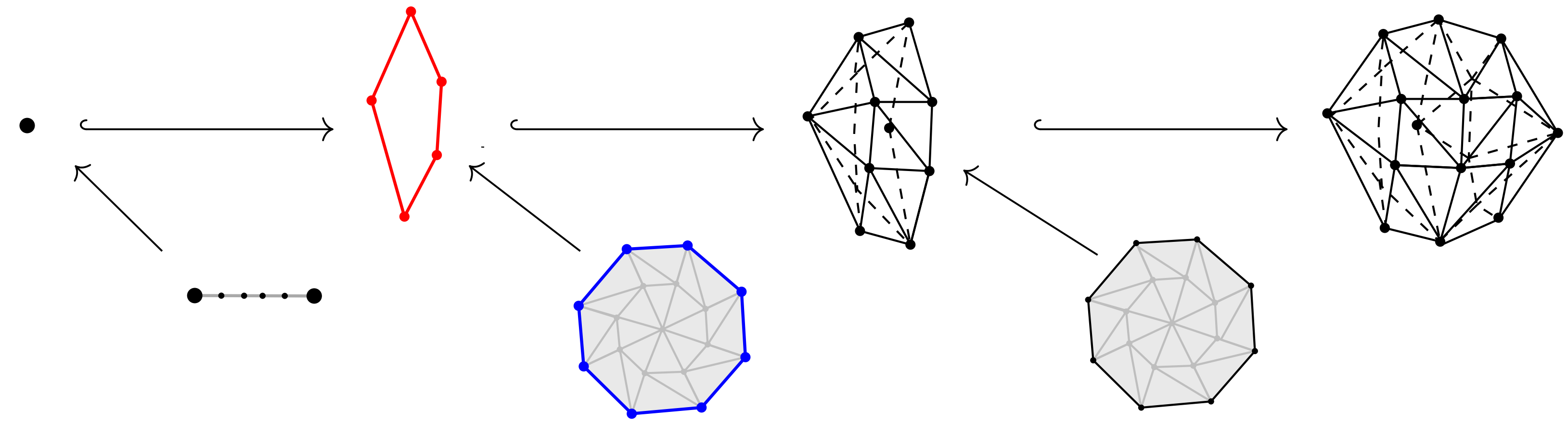
Problem: Refinements may increase the maximal diameter.



Proposition: After k Delaunay refinements, the **maximal circumradius** is at most α^k times the initial one, where

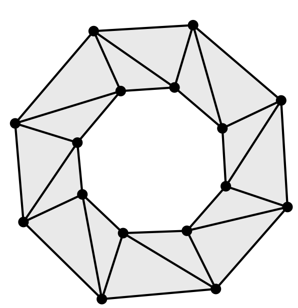
Steiner points	edge midpoints	minicenter	centroid
$\boldsymbol{\alpha}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$d/(d+1)$

Simplicial mapping cones

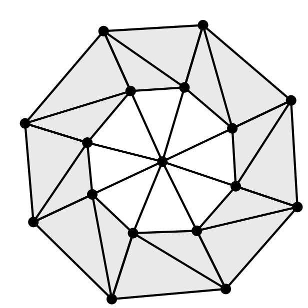


Say we found a simplicial map $g\colon K \to L$.

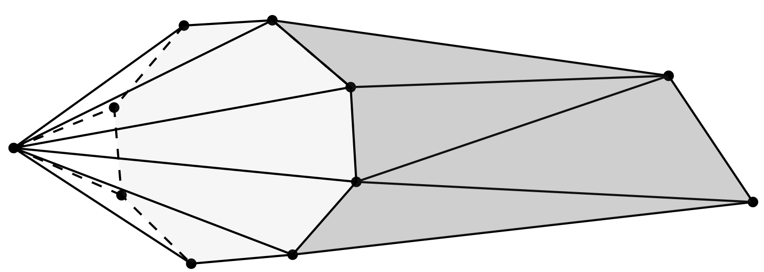
Next step: build the **mapping cone** of g.



Triangulate $|K| \times [0,1]$

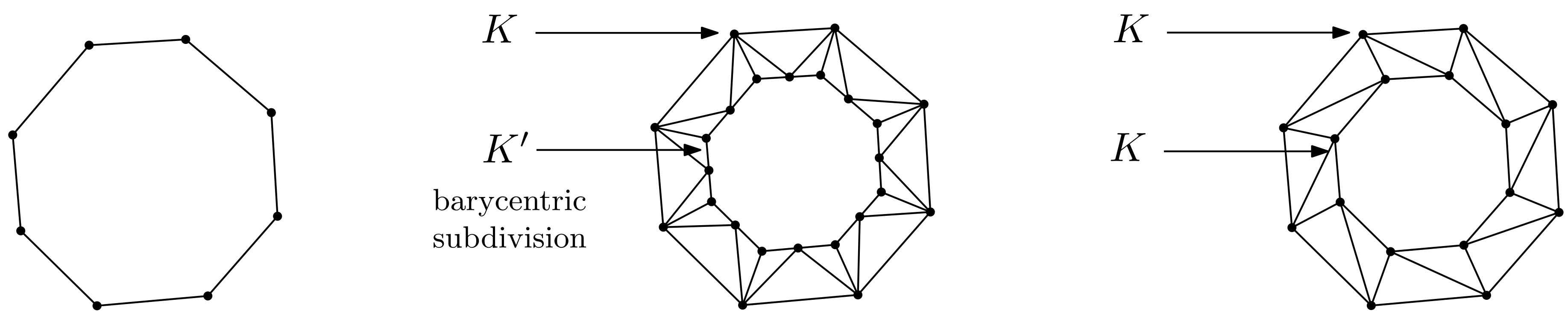


Cone the inner layer



Glue the outer layer on L

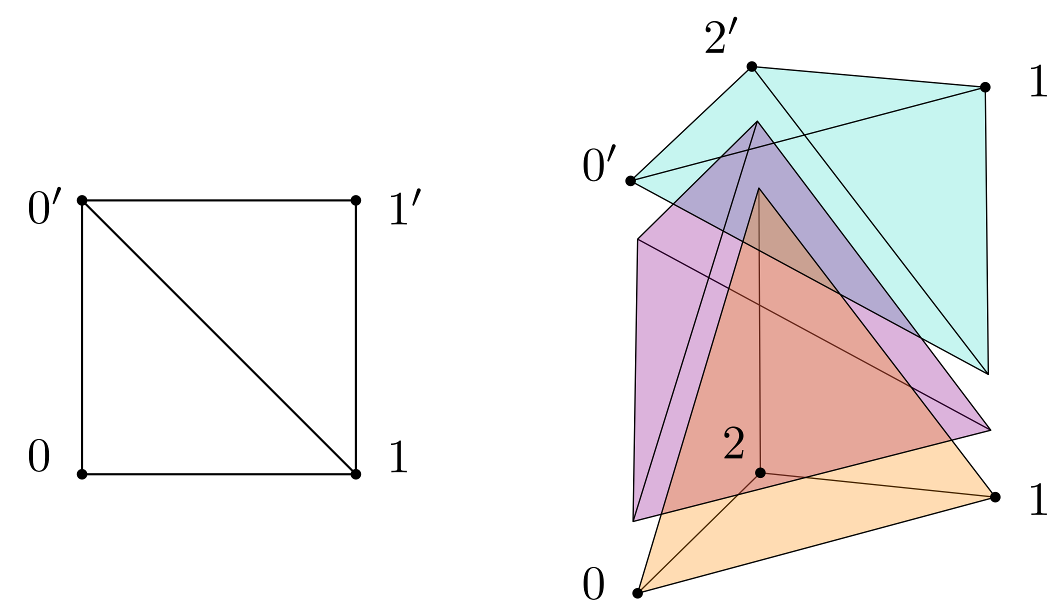
Triangulation of $|K| \times [0,1]$.



K (Cohen, 1967) Staircase triangulation

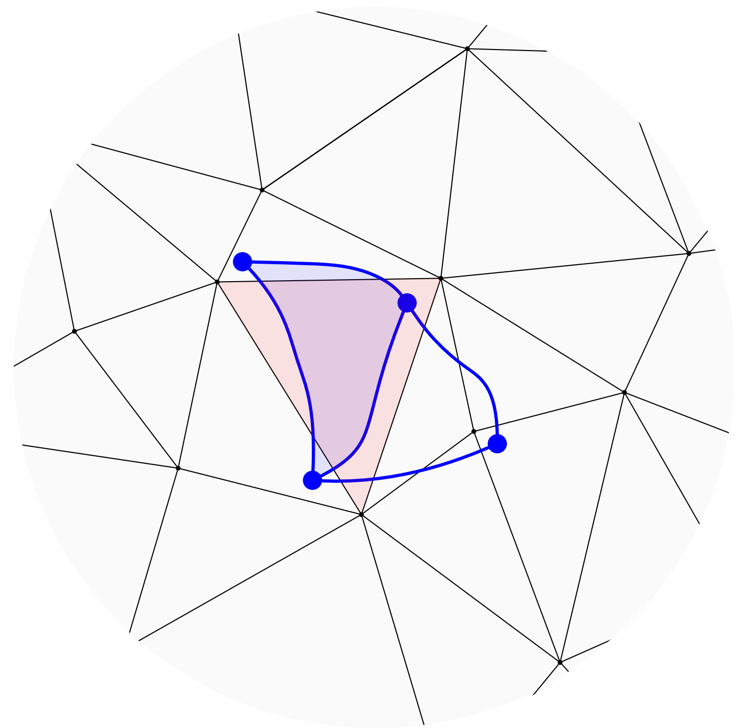
Staircase triangulation of $|\sigma| \times [0,1]$:

- Order the vertices $\{v_0, \ldots, v_d\}$,
- Take a copy $\{v'_0, \ldots, v'_d\}$,
- Insert $\sigma_k = [v_k, \ldots, v_d, v'_0, \ldots, v'_k]$ for $k \in [\![0, d]\!]$.



```
--- Triangulation of RP^3 ---
--- Init cell of dimension 0 ---
| Triangulation of 0-cell        | Dim/Verts/Facets/Splx = 0/1/0/1. Cell #0 initialized.
--- Glue cell of dimension 1 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 1/3/3/6. Cell #1 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 0 ] ]. Duration 0:04.042.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 3, 3 ]. Duration 0:01.776.
--- Glue cell of dimension 2 ---
| Spherical Delaunay             | Generate 10 points. Duration 0:00.000. Build hull. Duration 0:00.000. Dim/Verts/Facets = 1/10/10.
| Locate in triangulation        | Vertex 10/10. Duration 0:00.001.
| Check star condition           | Condition not satisfied for 40.0% of the vertices (4/10). Duration 0:00.000.
| Delaunay refinement            | Blame 4 0-simplices. Add 8 centroids from scratch. Duration 0:00.000. Min dist 3.1e-01.
| Locate in triangulation        | Vertex 8/8. Duration 0:00.001.
| Check star condition           | Condition satisfied. Duration 0:00.000.
| Triangulation of sphere        | Dim/Verts/Facets/Splx = 1/18/18/36. Min dist 3.1e-01. Mesh ratio 5.0e-01.
| Triangulation of ball          | Dim/Verts/Facets/Splx = 2/37/54/181. Min dist 1.6e-01.
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 2/22/42/127. Cell #2 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [  ] ]. Duration 0:02.147.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 22, 63, 42 ]. Duration 0:01.328.
--- Glue cell of dimension 3 ---
| Spherical Delaunay             | Generate 100 points. Duration 0:00.000. Build hull. Duration 0:00.001. Dim/Verts/Facets = 2/100/196.
| Locate in triangulation        | Vertex 100/100. Duration 0:00.028.
| Check star condition           | Condition not satisfied for 53.0% of the vertices (53/100). Duration 0:00.000.
| Delaunay refinement            | Blame 53 0-simplices. Add 153 centroids from scratch. Duration 0:00.005. Min dist 1.2e-01.
| Locate in triangulation        | Vertex 153/153. Duration 0:00.041.
| Check star condition           | Condition not satisfied for 30.83% of the vertices (78/253). Duration 0:00.000.
| Delaunay refinement            | Blame 78 0-simplices. Add 238 centroids from scratch. Duration 0:00.011. Min dist 8.3e-02.
| Locate in triangulation        | Vertex 238/238. Duration 0:00.054.
| Check star condition           | Condition not satisfied for 16.497% of the vertices (81/491). Duration 0:00.001.
| Delaunay refinement            | Blame 81 0-simplices. Add 281 centroids from scratch. Duration 0:00.017. Min dist 4.2e-02.
| Locate in triangulation        | Vertex 281/281. Duration 0:00.063.
| Check star condition           | Condition not satisfied for 3.497% of the vertices (27/772). Duration 0:00.001.
| Delaunay refinement            | Blame 27 0-simplices. Add 154 centroids from scratch. Duration 0:00.019. Min dist 3.2e-02.
| Locate in triangulation        | Vertex 154/154. Duration 0:00.033.
| Check star condition           | Condition satisfied. Duration 0:00.001.
| Triangulation of sphere        | Dim/Verts/Facets/Splx = 2/926/1848/5546. Min dist 3.2e-02. Mesh ratio 6.5e-02.
| Triangulation of ball          | Dim/Verts/Facets/Splx = 3/1853/7392/35121. Min dist 1.6e-02.
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 3/949/4493/19804. Cell #3 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [  ], [ 0 ] ]. Duration 0:03.494.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: false. Manifold: n/a. f-vector: [ 949, 5409, 8953, 4493 ]. Duration 0:01.501.
```

The map $f\colon |K| \to |L|$ satisfies the **star condition** if $\forall v \in V(K)$, $\exists w \in V(L)$ s.t. $f\left(\left|\overline{\mathrm{St}}\,(v)\right|\right) \subseteq |\mathrm{St}\,(w)|$.



Let g be a simplicial approximation.

If g maps a facet $\sigma \in K$ to a facet $\tau \in L$, then $f(|\sigma|) \subset |\tau|$.

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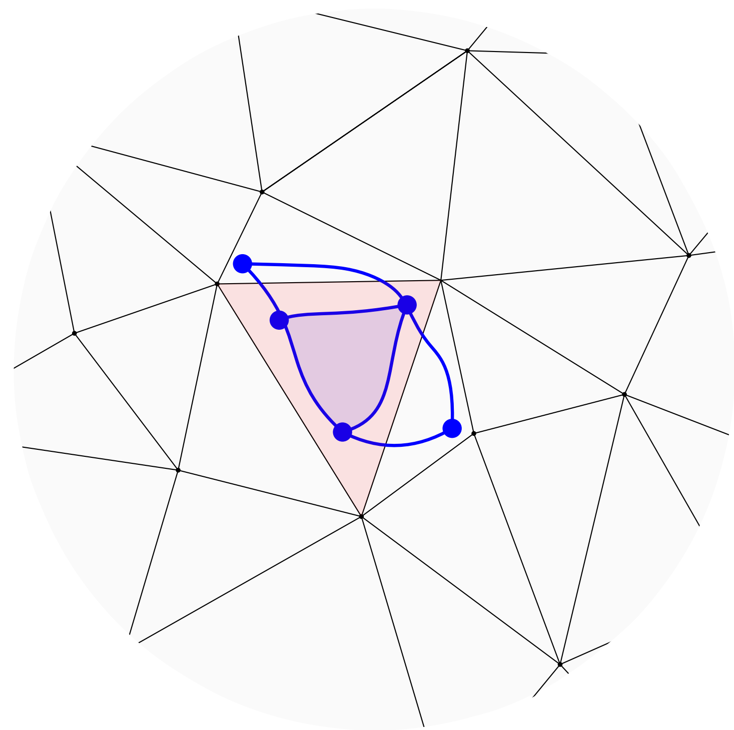


Let g be a simplicial approximation.

If g maps a facet $\sigma \in K$ to a facet $\tau \in L$, then $f(|\sigma|) \subset |\tau|$.

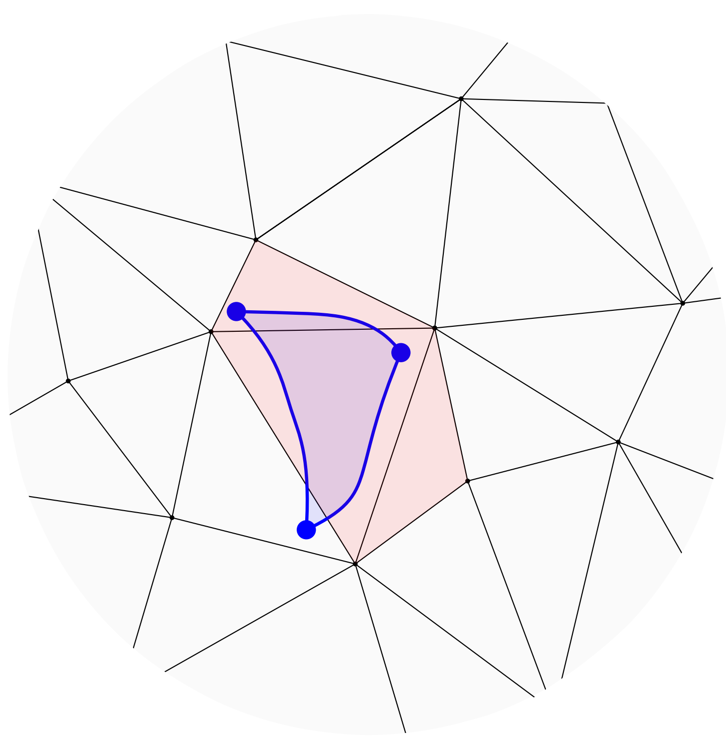
Each facet of L requires $\dim(L) + 1$ vertices in K.

Split simplicial approximation into two problems:

Geometric feasibility

For each facet $\sigma \in K$, find admissible facets in L.

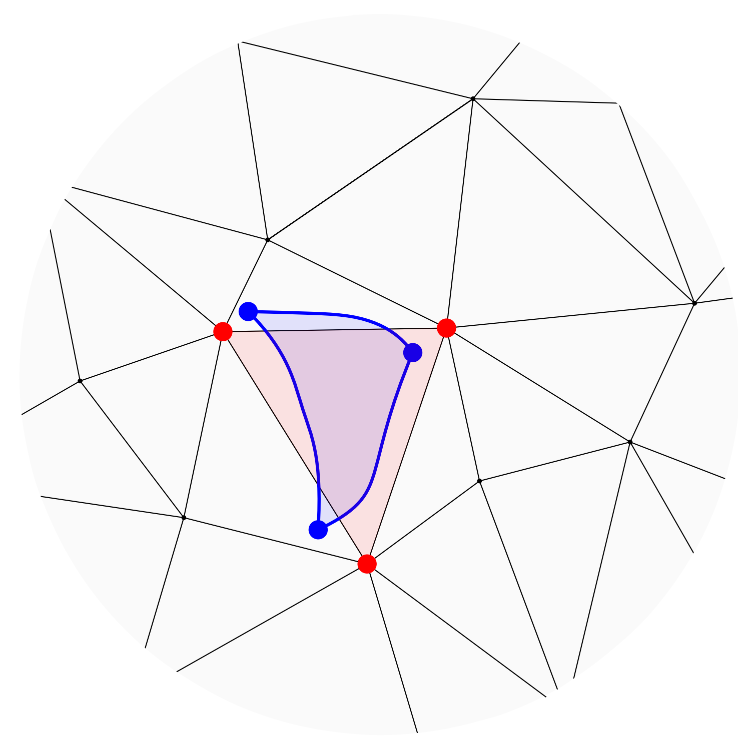
$\rightarrow$ Define a homotopy via equiconnecting maps.



Combinatorial feasibility

Given admissible facets, find a simplicial map.

$\rightarrow$ Solve the list homomorphism problem.



(Dugundji, 1965)

Idea: Draw paths $t \mapsto \Pi(x,y,t)$ in the space.

A space Y is **locally equiconnected** if there exists a neighborhood $U \subset Y \times Y$ of the diagonal and a continuous map $\Pi\colon U \times [0,1] \to Y$ such that for all $x, y \in U$ and $t \in [0,1]$,

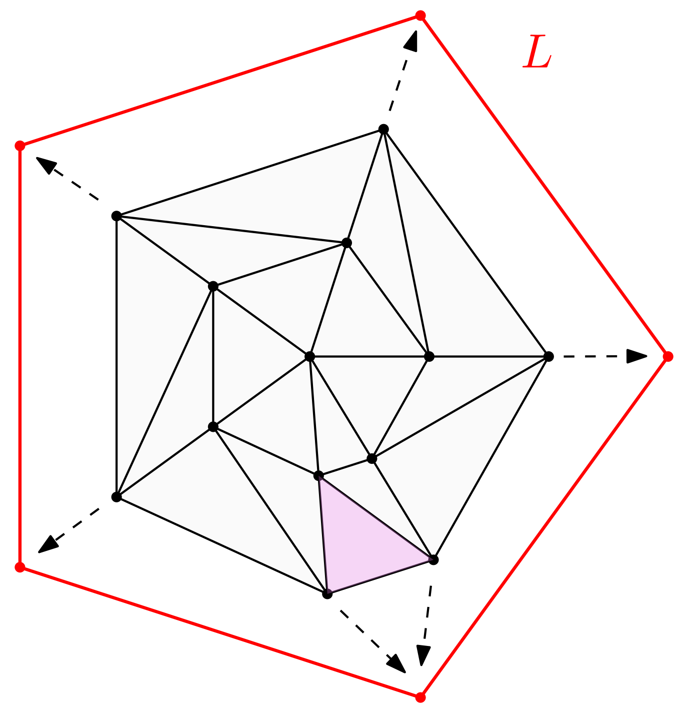
- $\Pi(x,y,0) = x$,
- $\Pi(x,y,1) = y$,
- $\Pi(x,x,t) = x$.

Observation: Two maps $f, g\colon X \to Y$ are homotopic whenever $(f(x), g(x)) \in U$ for all $x \in X$.
A homotopy is given by $H(x,t) = \Pi(f(x), g(x), t)$.

Theorem (Dyer, Eilenberg, 1972): A CW complex is locally equiconnected.

Let $g\colon K \to L$ simplicial, and $B(K)$ the triangulated ball.

The **simplicial gluing** $L \cup_g B(K)$ is different from the **standard gluing** $|L| \cup_{|g|} |B(K)|$.

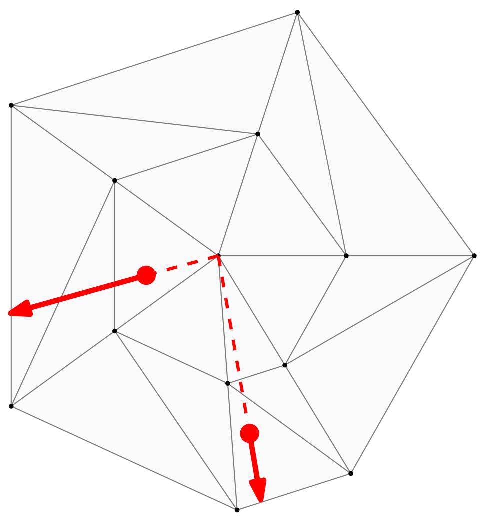


Proposition: After quotient, $B(K)$ admits on its interior an equiconnecting map $\Pi\colon U \times [0,1] \to |L \cup_g B(K)|$ such that $(x, y) \in U$ provided that

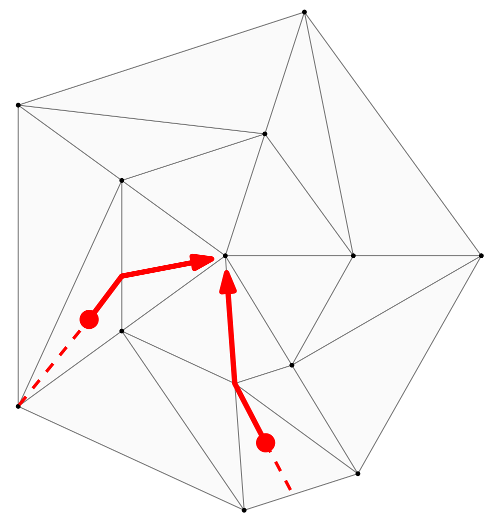
- $\|x\| = 0$,
- or $\|x\|, \|y\| \leq 1/2$,
- or $x/\|x\| \neq -y/\|y\|$.

Equiconnecting map on the triangulated ball

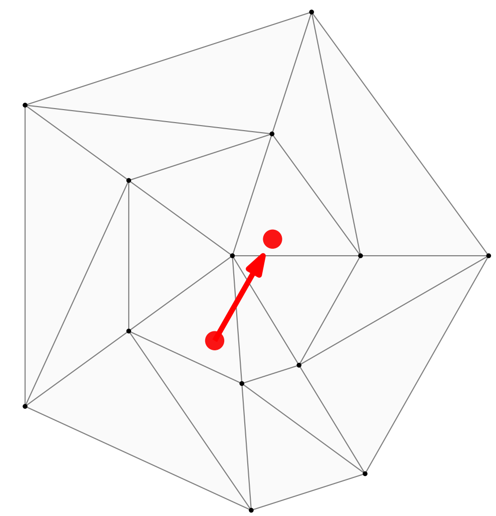
Admissible paths (that descend to the quotient).



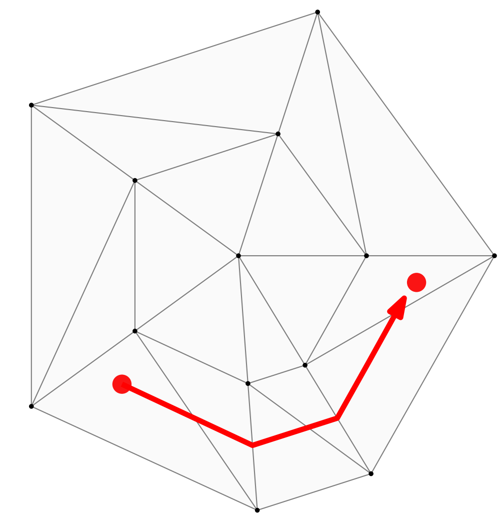
Ray away from the origin



Climb towards the origin

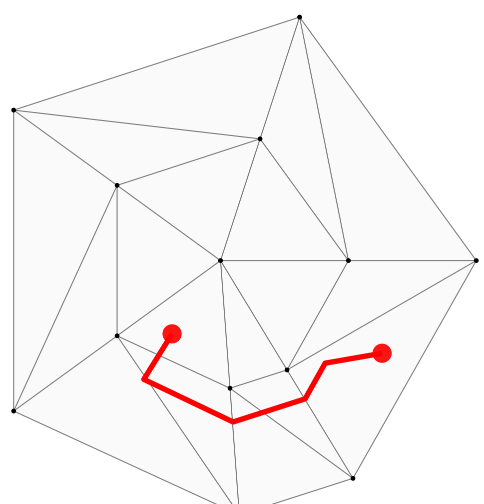


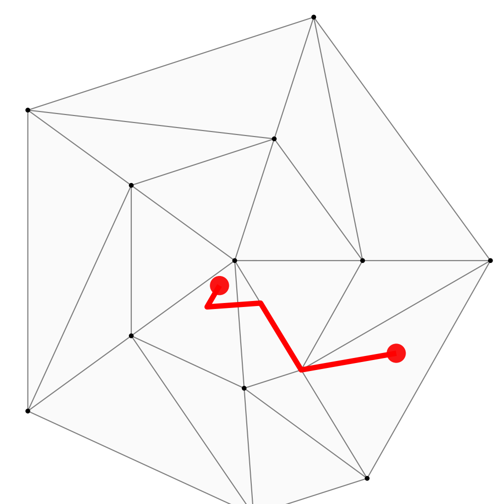
Straight path in inner layer



Circular arc in outer layer (points of equal norm)

They are combined to define paths on the ball.





Input: Simplicial complexes K, L and a subset $\ell(v) \subset V(L)$ for all $v \in V(K)$.

Output: A simplicial map $g\colon K \to L$ with $g(v) \in \ell(v)$ for all $v \in V(K)$.

Polynomial-time solvable if L is a bi-arc graph, NP-complete otherwise (Feder, Hell, Huang, 2003).

Software: `MiniSAT`, `OR-Tools`, `Z3`, ...

If the problem is not feasible, we solve the intermediary problem:

Output: The minimal number of facets of K to drop such that $K' \to L$ admits a solution.

Triangulation of $\mathbb{R}P^3$

```
 --- Triangulation of RP^3 ---
 --- Init cell of dimension 0 ---
| Triangulation of 0-cell        | Dim/Verts/Facets/Splx = 0/1/0/1. Cell #0 initialized.
 --- Glue cell of dimension 1 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 1/3/3/6. Cell #1 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 0 ] ]. Duration 0:02.257.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 3, 3 ]. Duration 0:01.362.
 --- Glue cell of dimension 2 ---
| Spherical Delaunay             | Generate 10 points. Duration 0:00.000. Build hull. Duration 0:00.000. Dim/Verts/Facets = 1/10/10.
| Locate in triangulation        | Vertex 10/10. Duration 0:00.001.
| Check Lipschitz criterion      | Facet 10/10... Duration 0:00.002. Criterion satisfied.
| LHom (dim 1 without dropping)  | Get 1-skeleton. Duration 0:00.000. List vars. Duration 0:00.000. Solve. Duration 0:00.003. Problem feasible.
| Triangulation of sphere        | Dim/Verts/Facets/Splx = 1/10/10/20. Min dist 6.3e-01. Mesh ratio 1.0e+00.
| Triangulation of ball          | Dim/Verts/Facets/Splx = 2/21/30/101. Min dist 3.1e-01.
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 2/14/26/79. Cell #2 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [  ] ]. Duration 0:03.057.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 14, 39, 26 ]. Duration 0:01.347.
 --- Glue cell of dimension 3 ---
| Spherical Delaunay             | Generate 100 points. Duration 0:00.000. Build hull. Duration 0:00.000. Dim/Verts/Facets = 2/100/196.
| Locate in triangulation        | Vertex 100/100. Duration 0:00.028.
| Check Lipschitz criterion      | Facet 196/196... Duration 0:00.038. Criterion not satisfied for 1.02% of facets (2/196).
| Delaunay refinement            | Blame 2 facets. Add 2 centroids from scratch. Duration 0:00.002. Min dist 1.9e-01.
| Locate in triangulation        | Vertex 2/2. Duration 0:00.001.
| Check Lipschitz criterion      | Facet 200/200... Duration 0:00.003. Criterion satisfied.
| LHom (dim 1 without dropping)  | Get 1-skeleton. Duration 0:00.000. List vars. Duration 0:00.004. Solve. Duration 0:00.002. Problem not feasible.
| LHom (dim 1 with dropping)     | Get 1-skeleton. Duration 0:00.000. List vars. Duration 0:00.006. Solve. Duration 0:00.033. Dropped 1 simplices.
| Delaunay refinement            | Blame 1 1-simplices. Add 2 centroids from scratch. Duration 0:00.002. Min dist 1.6e-01.
| Locate in triangulation        | Vertex 2/2. Duration 0:00.001.
| Check Lipschitz criterion      | Facet 204/204... Duration 0:00.003. Criterion satisfied.
| LHom (dim 1 without dropping)  | Get 1-skeleton. Duration 0:00.000. List vars. Duration 0:00.010. Solve. Duration 0:00.005. Problem feasible.
| LHom (dim 2 without dropping)  | Get 2-skeleton. Duration 0:00.000. List vars. Duration 0:00.004. Solve. Duration 0:00.009. Problem feasible.
| Triangulation of sphere        | Dim/Verts/Facets/Splx = 2/104/204/614. Min dist 1.6e-01. Mesh ratio 2.9e-01.
| Triangulation of ball          | Dim/Verts/Facets/Splx = 3/209/816/3885. Min dist 8.0e-02.
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 3/119/593/2602. Cell #3 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [  ], [ 0 ] ]. Duration 0:02.244.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: false. Manifold: n/a. f-vector: [ 119, 708, 1182, 593 ]. Duration 0:01.339.
```

Triangulation of $\mathbb{R}P^3$ with Delaunay simplifications


```
--- Triangulation of RP^3 ---
--- Init cell of dimension 0 ---
| Triangulation of 0-cell       | Dim/Verts/Facets/Splx = 0/1/0/1. Cell #0 initialized.
--- Glue cell of dimension 1 ---
| Triangulation of mapping cone | Dim/Verts/Facets/Splx = 1/3/3/6. Cell #1 glued.
| Homology groups with <gap>    | [ [ 0 ], [ 0 ] ]. Duration 0:02.224.
| Is manifold with <gap>        | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 3, 3 ]. Duration 0:01.305.
--- Glue cell of dimension 2 ---
| Spherical Delaunay            | Generate 10 points. Duration 0:00.000. Build hull. Duration 0:00.001. Dim/Verts/Facets = 1/10/10.
| Locate in triangulation       | Vertex 10/10. Duration 0:00.002.
| Check Lipschitz criterion     | Facet 10/10... Duration 0:00.002. Criterion satisfied.
| LHom (dim 1 without dropping) | Get 1-skeleton. Duration 0:00.000. List vars. Duration 0:00.000. Solve. Duration 0:00.004. Problem feasible.
| Delaunay simplification       | Initialize. Duration 0:00.003. Vertex 5/10. Duration 0:00.002.
| Triangulation of sphere       | Dim/Verts/Facets/Splx = 1/6/6/12. Min dist 6.3e-01. Mesh ratio 5.0e-01.
| Triangulation of ball         | Dim/Verts/Facets/Splx = 2/13/18/61. Min dist 3.1e-01.
| Triangulation of mapping cone | Dim/Verts/Facets/Splx = 2/10/18/55. Cell #2 glued.
| Homology groups with <gap>    | [ [ 0 ], [ 2 ], [  ] ]. Duration 0:02.142.
| Is manifold with <gap>        | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 10, 27, 18 ]. Duration 0:01.309.
--- Glue cell of dimension 3 ---
| Spherical Delaunay            | Generate 100 points. Duration 0:00.000. Build hull. Duration 0:00.000. Dim/Verts/Facets = 2/100/196.
| Locate in triangulation       | Vertex 100/100. Duration 0:00.034.
| Check Lipschitz criterion     | Facet 196/196... Duration 0:00.039. Criterion not satisfied for 1.02% of facets (2/196).
| Delaunay refinement           | Blame 2 facets. Add 2 centroids from scratch. Duration 0:00.002. Min dist 1.9e-01.
| Locate in triangulation       | Vertex 2/2. Duration 0:00.001.
| Check Lipschitz criterion     | Facet 200/200... Duration 0:00.003. Criterion satisfied.
| LHom (dim 1 without dropping) | Get 1-skeleton. Duration 0:00.000. List vars. Duration 0:00.004. Solve. Duration 0:00.010. Problem feasible.
| LHom (dim 2 without dropping) | Get 2-skeleton. Duration 0:00.000. List vars. Duration 0:00.005. Solve. Duration 0:00.024. Problem feasible.
| Delaunay simplification       | Initialize. Duration 0:00.021. Vertex 80/102. Duration 0:00.214.
| Triangulation of sphere       | Dim/Verts/Facets/Splx = 2/23/42/128. Min dist 3.4e-01. Mesh ratio 1.6e-01.
| Triangulation of ball         | Dim/Verts/Facets/Splx = 3/47/168/807. Min dist 1.7e-01.
| Triangulation of mapping cone | Dim/Verts/Facets/Splx = 3/34/159/704. Cell #3 glued.
| Homology groups with <gap>    | [ [ 0 ], [ 2 ], [  ], [ 0 ] ]. Duration 0:02.167.
| Is manifold with <gap>        | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 34, 193, 318, 159 ]. Duration 0:01.372.
```

Triangulation of $\mathbb{R}P^4$ with Delaunay simplifications

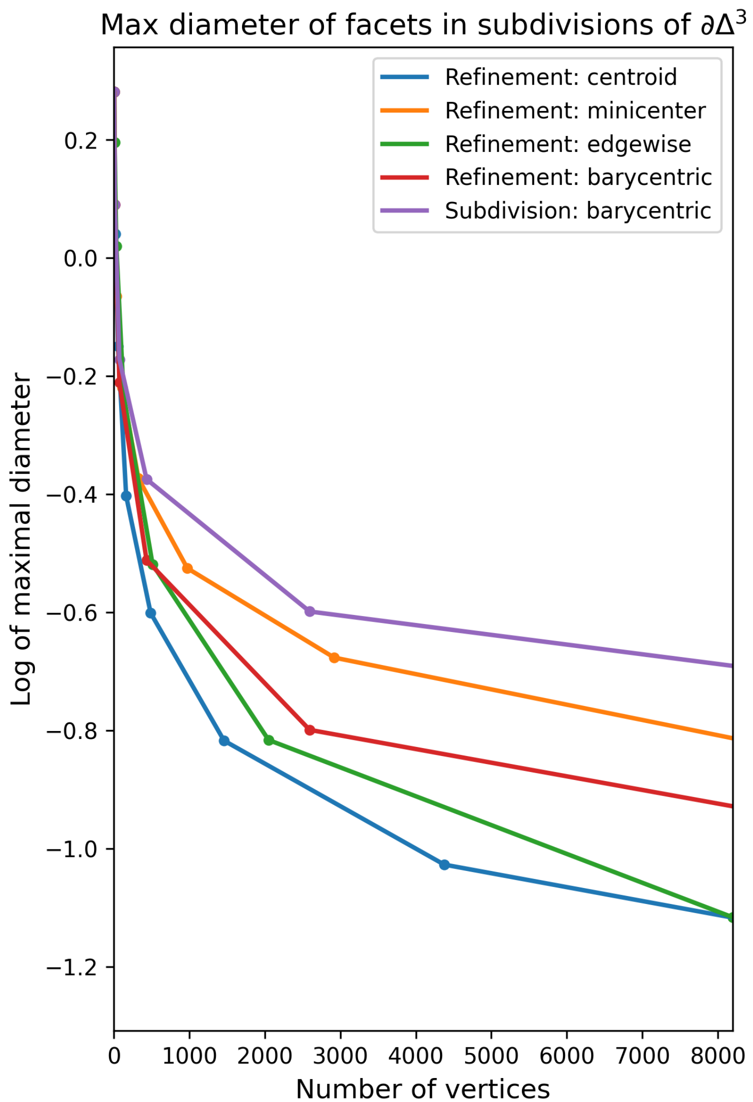
```
--- Glue cell of dimension 4 ---
| Spherical Delaunay              | Generate 500 points. Duration 0:01.277. Build hull. Duration 0:00.004. Dim/Verts/Facets = 3/500/2881.
| Locate in triangulation         | Vertex 500/500. Duration 0:00.349.
| Check Lipschitz criterion       | Facet 2881/2881... Duration 0:00.769. Criterion not satisfied for 6.734% of facets (194/2881).
| Delaunay refinement             | Blame 194 facets. Add 194 centroids from scratch. Duration 0:00.045. Min dist 1.0e-01.
| Locate in triangulation         | Vertex 194/194. Duration 0:00.114.
| Check Lipschitz criterion       | Facet 4126/4126... Duration 0:00.504. Criterion not satisfied for 1.018% of facets (42/4126).
| Delaunay refinement             | Blame 42 facets. Add 42 centroids from scratch. Duration 0:00.048. Min dist 6.5e-02.
| Locate in triangulation         | Vertex 42/42. Duration 0:00.031.
| Check Lipschitz criterion       | Facet 4385/4385... Duration 0:00.197. Criterion not satisfied for 0.023% of facets (1/4385).
| Delaunay refinement             | Blame 1 facets. Add 1 centroids from scratch. Duration 0:00.049. Min dist 6.5e-02.
| Locate in triangulation         | Vertex 1/1. Duration 0:00.001.
| Check Lipschitz criterion       | Facet 4389/4389... Duration 0:00.032. Criterion satisfied.
| LHom (dim 1 without dropping)   | Get 1-skeleton. Duration 0:00.012. List vars. Duration 0:00.091. Solve. Duration 0:00.012. Problem not feasible.
| LHom (dim 1 with dropping)      | Get 1-skeleton. Duration 0:00.012. List vars. Duration 0:00.165. Solve. Duration 0:02.617. Dropped 8 simplices.
| Delaunay refinement             | Blame 8 1-simplices. Add 41 centroids from scratch. Duration 0:00.050. Min dist 6.5e-02.
| Locate in triangulation         | Vertex 41/41. Duration 0:00.012.
| Check Lipschitz criterion       | Facet 4649/4649... Duration 0:00.158. Criterion not satisfied for 0.022% of facets (1/4649).
| Delaunay refinement             | Blame 1 facets. Add 1 centroids from scratch. Duration 0:00.053. Min dist 6.5e-02.
| Locate in triangulation         | Vertex 1/1. Duration 0:00.001.
| Check Lipschitz criterion       | Facet 4654/4654... Duration 0:00.034. Criterion satisfied.
| LHom (dim 1 without dropping)   | Get 1-skeleton. Duration 0:00.013. List vars. Duration 0:00.105. Solve. Duration 0:00.096. Problem feasible.
| LHom (dim 2 without dropping)   | Get 2-skeleton. Duration 0:00.010. List vars. Duration 0:00.386. Solve. Duration 0:00.085. Problem not feasible.
| LHom (dim 2 with dropping)      | Get 2-skeleton. Duration 0:00.010. List vars. Duration 0:00.440. Solve. Duration 0:11.208. Dropped 3 simplices.
| Delaunay refinement             | Blame 3 2-simplices. Add 3 centroids from scratch. Duration 0:00.128. Min dist 6.5e-02.
| Locate in triangulation         | Vertex 3/3. Duration 0:00.002.
| Check Lipschitz criterion       | Facet 4678/4678... Duration 0:00.099. Criterion satisfied.
| LHom (dim 1 without dropping)   | Get 1-skeleton. Duration 0:00.029. List vars. Duration 0:00.238. Solve. Duration 0:00.164. Problem feasible.
| LHom (dim 2 without dropping)   | Get 2-skeleton. Duration 0:00.014. List vars. Duration 0:00.512. Solve. Duration 0:02.512. Problem feasible.
| LHom (dim 3 without dropping)   | Get 3-skeleton. Duration 0:00.003. List vars. Duration 0:00.661. Solve. Duration 0:02.948. Problem feasible.
| Delaunay simplification         | Initialize. Duration 0:00.577. Vertex 587/782. Duration 0:22.182.
| Triangulation of sphere         | Dim/Verts/Facets/Splx = 3/196/1121/4876. Min dist 1.2e-01. Mesh ratio 7.6e-02.
| Triangulation of ball           | Dim/Verts/Facets/Splx = 4/393/5605/37833. Min dist 5.9e-02.
| Triangulation of mapping cone   | Dim/Verts/Facets/Splx = 4/231/4131/26147. Cell #4 glued.
| Homology groups with <gap>      | [ [ 0 ], [ 2 ], [  ], [ 2 ], [  ] ]. Duration 0:03.945.
| Is manifold with <gap>          | Pure: true. Pseudo-manifold: false. Manifold: n/a. f-vector: [ 231, 2751, 8712, 10322, 4131 ]. Duration 0:01.514.
```

Triangulation of $\mathcal{G}(2, \mathbb{R}^4)$

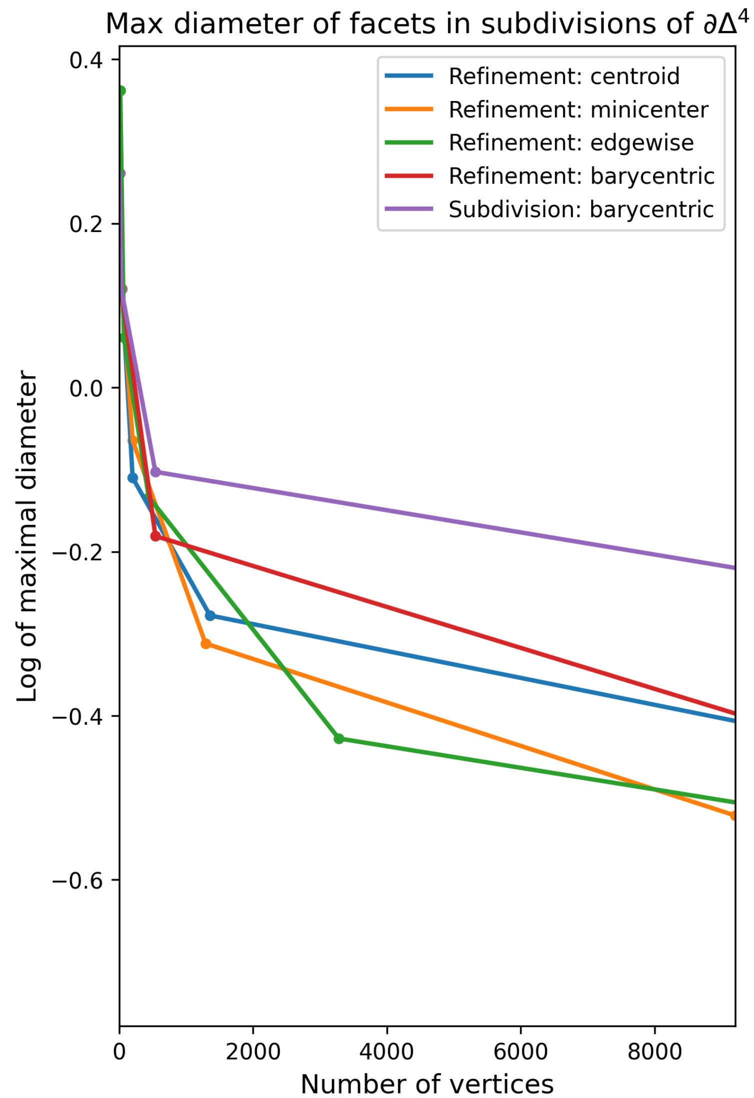

```
------ Triangulate G(2,4) ------
--- Init cell of dimension 0 ---
| Triangulation of 0-cell        | Dim/Verts/Facets/Splx = 0/1/0/1. Cell #0 initialized.
--- Glue cell of dimension 1 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 1/3/3/6. Cell #1 glued.
--- Glue cell of dimension 2 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 2/10/18/55. Cell #2 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [  ] ]. Duration 0:02.121.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: true. Manifold: true. f-vector: [ 10, 27, 18 ]. Duration 0:01.308.
--- Glue cell of dimension 2 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 2/17/36/104. Cell #3 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [ 0 ] ]. Duration 0:02.156.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: false. Manifold: n/a. f-vector: [ 17, 51, 36 ]. Duration 0:01.321.
--- Glue cell of dimension 3 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 3/73/376/1649. Cell #4 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [ 2 ], [  ] ]. Duration 0:02.188.
| Is manifold with <gap>         | Pure: true. Pseudo-manifold: true. Manifold: false. f-vector: [ 73, 448, 752, 376 ]. Duration 0:01.493.
--- Glue cell of dimension 4 ---
| Triangulation of mapping cone  | Dim/Verts/Facets/Splx = 4/879/16851/106044. Cell #5 glued.
| Homology groups with <gap>     | [ [ 0 ], [ 2 ], [ 2 ], [  ], [ 0 ] ]. Duration 0:15.182.
| Is manifold with <gap>         | Pure: false. Pseudo-manifold: n/a. Manifold: n/a. f-vector: [ 879, 10980, 35294, 42041, 16850 ]. Duration 0:02.845.
```

Thanks!

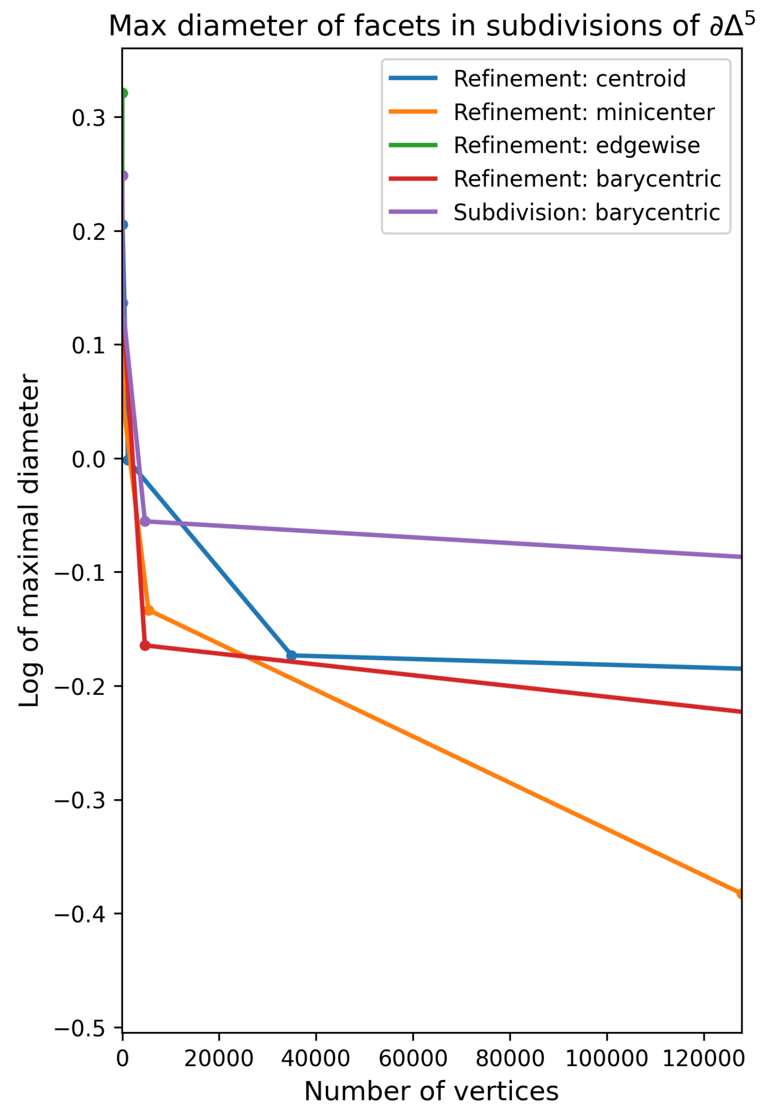
Subdivisions of S^2

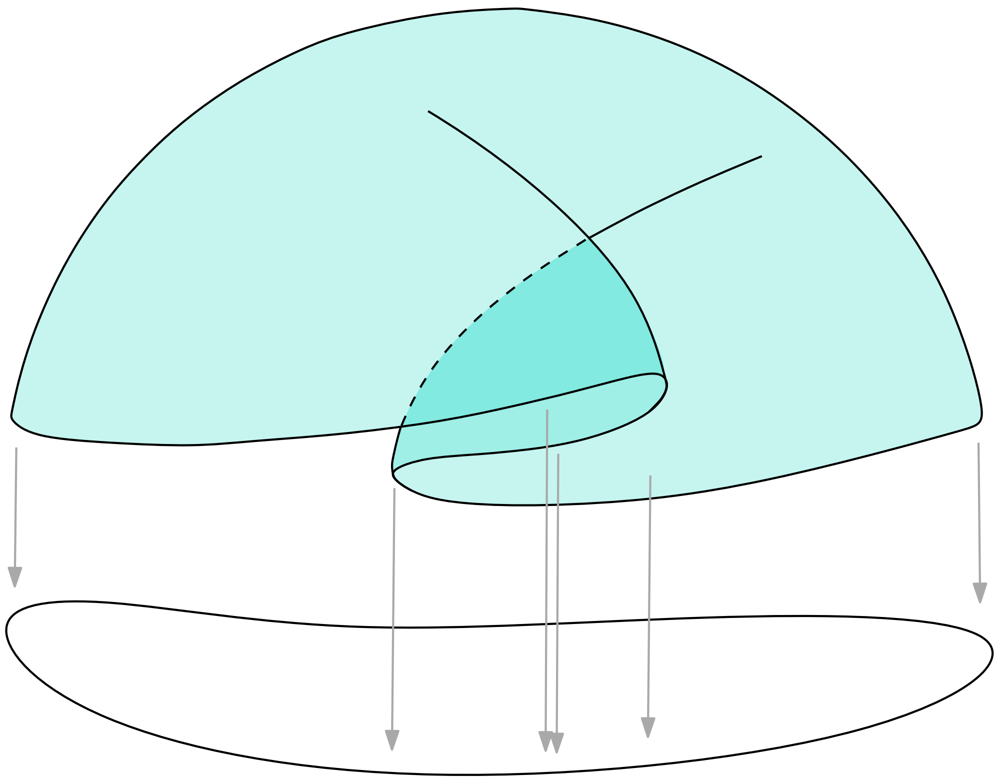


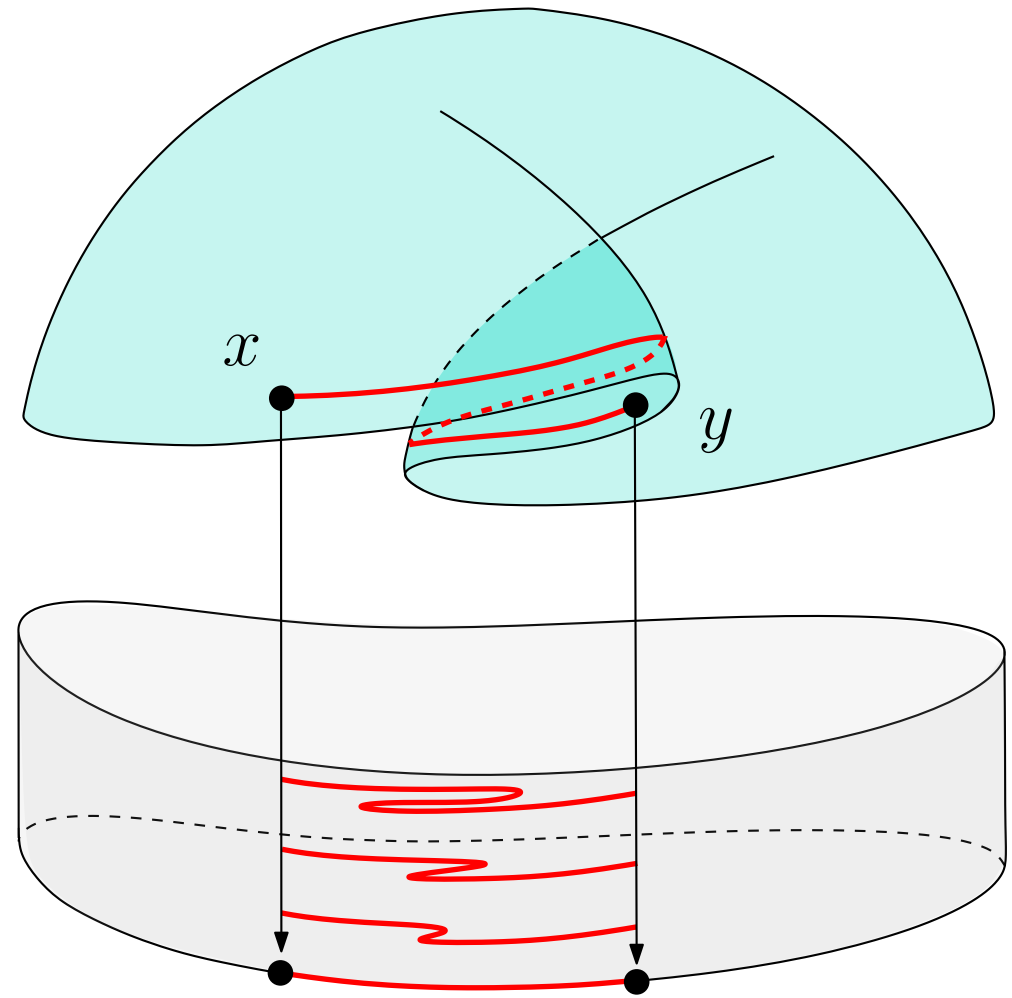
Subdivisions of S^3



Subdivisions of S^4






x
y