Dynamical Systems & Applications — 28/09/23

An introduction to Topological Data Analysis

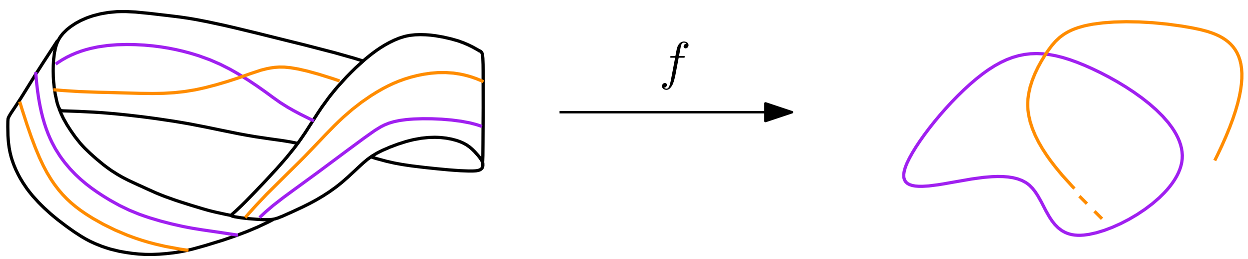
Part II/IV: Homology & homological inference

https://raphaeltinarrage.github.io

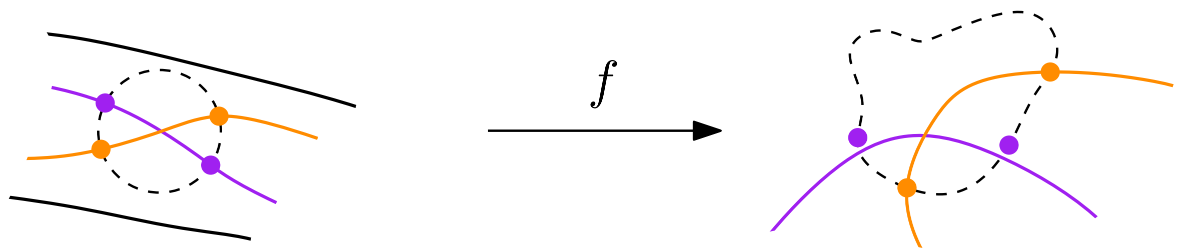
Last update: September 28, 2023

Sobre o último curso

Non-embedabbility of the Möbius strip: Suppose we have a embedding f into $\mathbb{R}^2$.
We draw two circles on the strip, C_1 and C_2, that only intersect once.



In $\mathbb{R}^2$, the circles C^1 and C^2 only intersect once, contradicting Jordan's theorem.
The circles **are not tangent** because of the following drawing (the order of the points is respected):



Homework: The intervals $[0, 1)$ and $(0, 1)$ are not homeomorphic.

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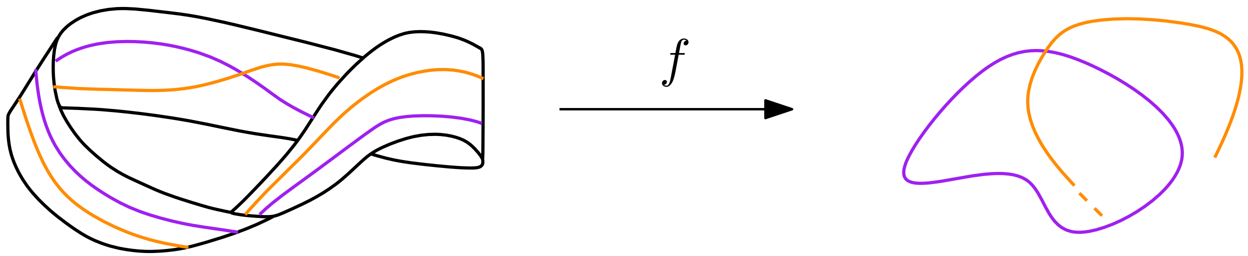

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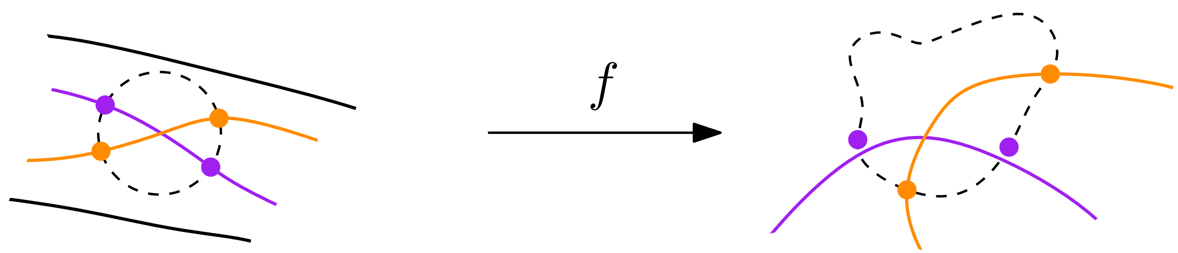


Homework: The intervals $[0,1)$ and $(0,1)$ are not homeomorphic.

By contradiction, suppose there exists a homeomorphism $f\colon [0,1) \to (0,1)$. Consider the induced map $g\colon [0,1)\setminus\{0\} \to (0,1)\setminus\{f(0)\}$. The map g is a homeomorphism. But $[0,1)\setminus\{0\}$ has one connected component, and $(0,1)\{f(0)\}$ two. This is impossible.

Part I/IV: Topological invariants

Tuesday 26th

Part II/IV: Homology

Thursday 28th

Part III/IV: Persistent Homology

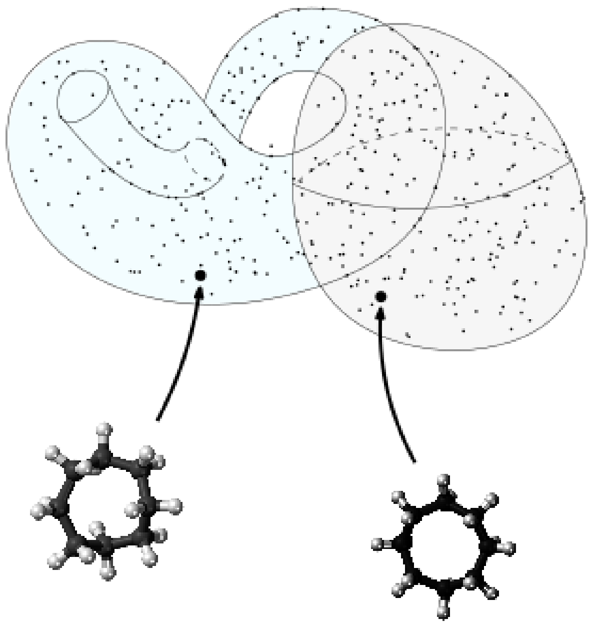
Tuesday 3rd

Part IV/IV: Python tutorial

Thursday 5th

Lembrete de ontem

Some datasets contain topology

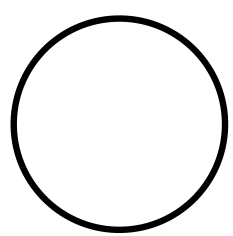
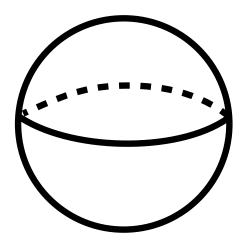
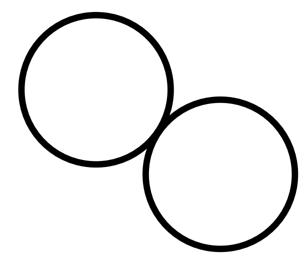
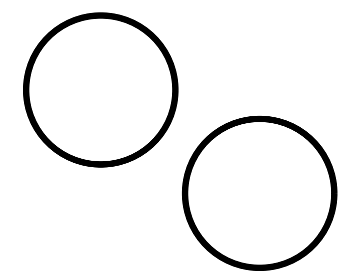


Invariants of homotopy classes allow to describe and understand topological spaces

Number of connected components

Euler characteristic χ

Betti numbers $\beta_0, \beta_1, \beta_2, \ldots$

X					
$\beta_0(X)$	1	1	1	1	2
$\beta_1(X)$	0	1	0	2	2
$\beta_2(X)$	0	0	1	0	0

Cardápio

Today we will define a powerful invariant, **homology groups**, that already contains the number of connected components, and the Euler characteristic.

$\boxed{\text{Algebraic topology}}$

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$$\begin{tikzcd} & & \boxed{\text{Algebraic topology}} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\ \text{Homology groups} & \text{Cohomology algebra} & \text{Characteristic classes} & \text{Functor } K \end{tikzcd}$$

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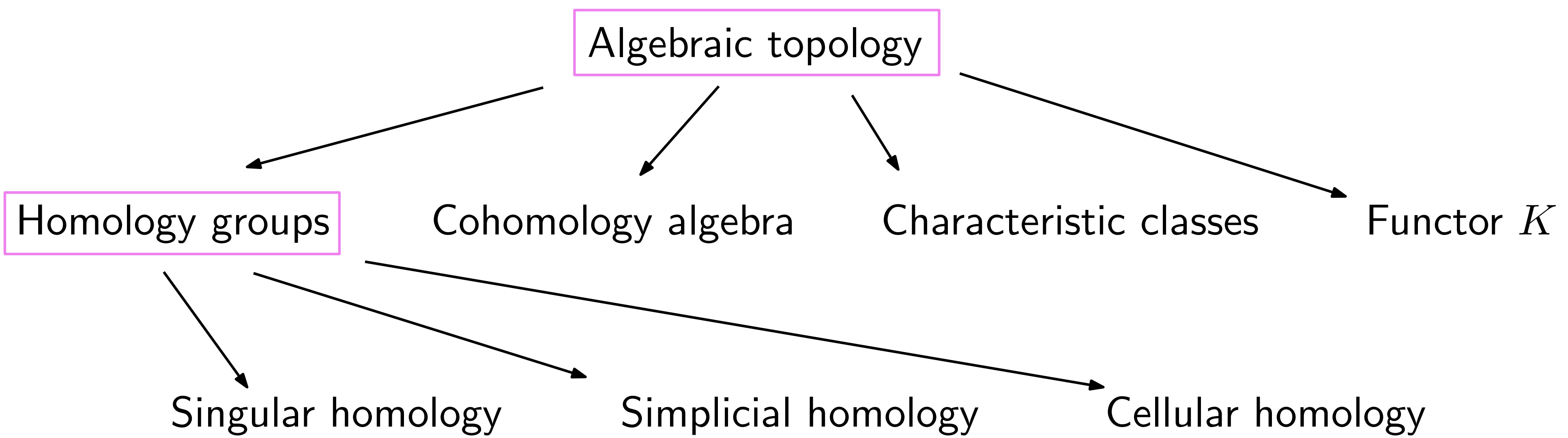

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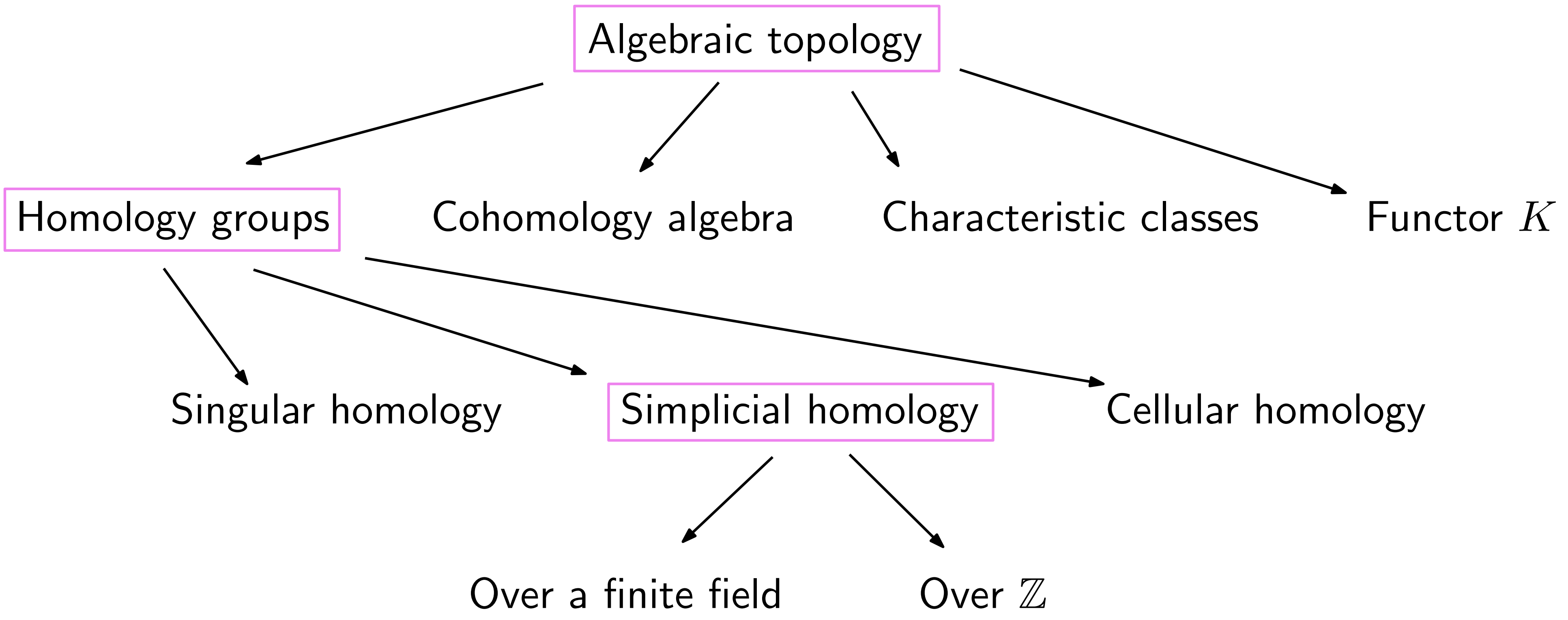

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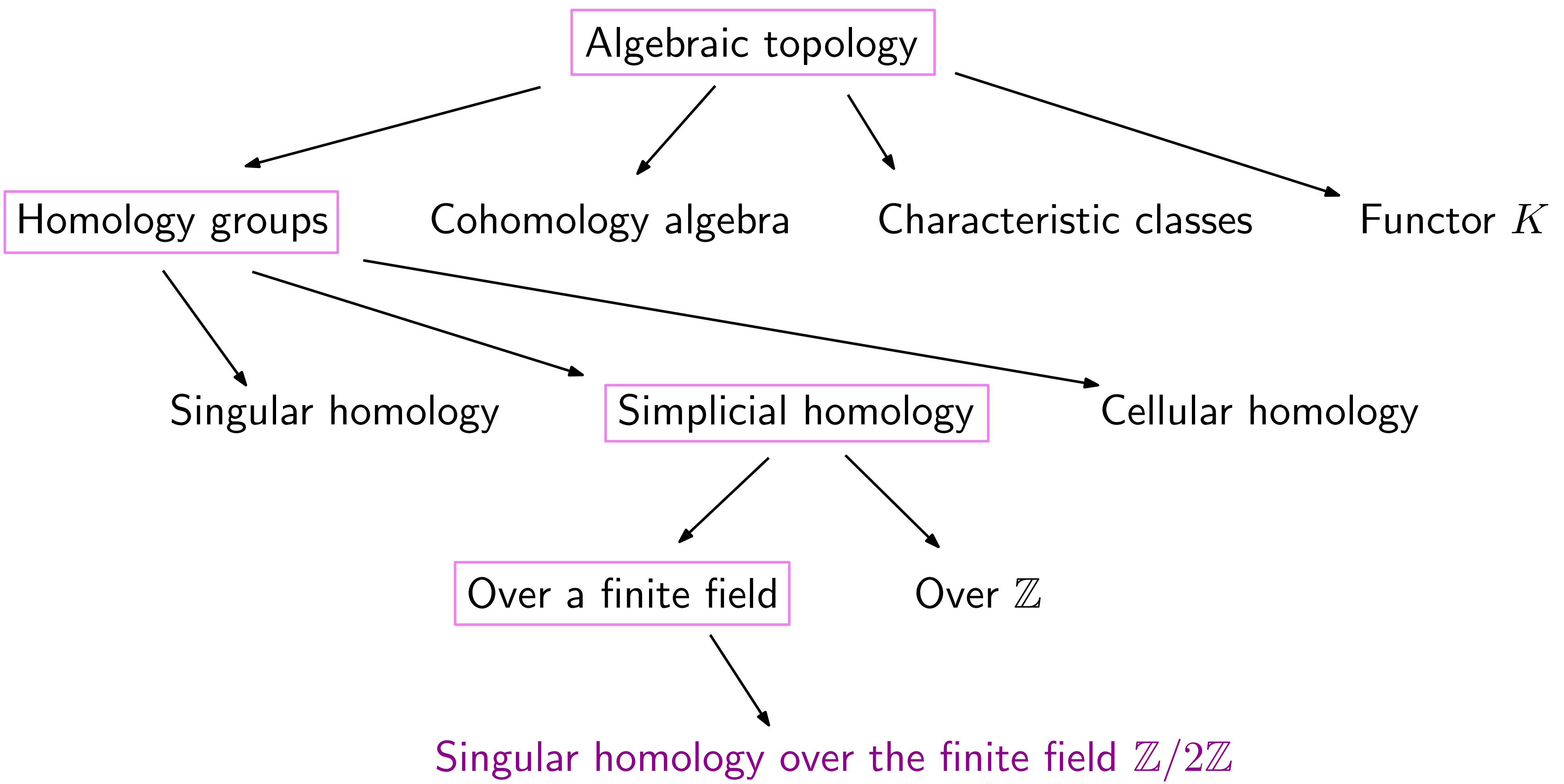

I - Simplicial homology

1 - Reminder of algebra

2 - Homological algebra

3 - Incremental algorithm

II - More about homology

1 - Topology of simplicial complexes

2 - Singular homology

3 - Functoriality

III - Homological inference

1 - Thickening parameter selection

2 - Čech complex

3 - Rips complex

O grupo $\mathbb{Z}/2\mathbb{Z}$

The **group** $\mathbb{Z}/2\mathbb{Z}$ can be seen as the set $\{0, 1\}$ with the operation

$$\begin{aligned} 0 + 0 &= 0 \\ 0 + 1 &= 1 \\ 1 + 0 &= 1 \\ 1 + 1 &= 0 \end{aligned}$$

For any $n \geq 1$, the **product group** $(\mathbb{Z}/2\mathbb{Z})^n$ is the group whose underlying set is

$$(\mathbb{Z}/2\mathbb{Z})^n = \{(\epsilon_1, ..., \epsilon_n),\ \epsilon_1, ..., \epsilon_n \in \mathbb{Z}/2\mathbb{Z}\}$$

and whose operation is defined as

$$(\epsilon_1, ..., \epsilon_n) + (\epsilon'_1, ..., \epsilon'_n) = (\epsilon_1 + \epsilon'_1, ..., \epsilon_n + \epsilon'_n).$$

The group $\mathbb{Z}/2\mathbb{Z}$ can be given a **field** structure

$$\begin{aligned} 0 \times 0 &= 0 \\ 0 \times 1 &= 0 \\ 1 \times 0 &= 0 \\ 1 \times 1 &= 1 \end{aligned}$$

and $(\mathbb{Z}/2\mathbb{Z})^n$ can be seen as a $\mathbb{Z}/2\mathbb{Z}$-**vector space** over the field $\mathbb{Z}/2\mathbb{Z}$.

Definition: A **vector space** over $\mathbb{Z}/2\mathbb{Z}$ is a set V endowed with two operations

$$\begin{aligned} V \times V &\longrightarrow V \\ (u,v) &\longmapsto u+v \end{aligned} \qquad \begin{aligned} \mathbb{Z}/2\mathbb{Z} \times V &\longrightarrow V \\ (\lambda,v) &\longmapsto \lambda \cdot v \end{aligned}$$

such that

(associativity) $\forall u,v,w \in V,\ \ (u+v)+w = u+(v+w)$,
(identity) $\exists 0 \in V$, $\forall v \in V,\ \ v+0 = 0+v = v$,
(inverse) $\forall v \in V, \exists w \in V,\ \ u+v = v+u = 0$,
(commutativity) $\forall u,v \in V,\ \ u+v = v+u$,
(compatibility of multiplication) $\forall \lambda, \mu \in \mathbb{Z}/2\mathbb{Z}, \forall v \in V, \lambda \cdot (\mu \cdot v) = (\lambda \times \mu) \cdot v$,
(scalar identity) $\forall v \in V, 1 \cdot v = v$,
(scalar distributivity) $\forall \mu, \nu \in \mathbb{Z}/2\mathbb{Z}, \forall v \in V,\ (\lambda + \nu) \cdot v = \lambda \cdot v + \nu \cdot v$,
(vector distributivity) $\forall \mu \in \mathbb{Z}/2\mathbb{Z}, \forall v, w \in V,\ \lambda \cdot (u+v) = \lambda \cdot v + \nu \cdot v$.

Definition: A **vector space** over $\mathbb{Z}/2\mathbb{Z}$ is a set V endowed with two operations

$$V \times V \longrightarrow V \qquad\qquad \mathbb{Z}/2\mathbb{Z} \times V \longrightarrow V$$
$$(u,v) \longmapsto u+v \qquad\qquad (\lambda, v) \longmapsto \lambda \cdot v$$

such that

$$\begin{array}{rl}
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\end{array}$$

Proposition: Le $(V,+)$ be a commutative group.
It can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure iff $\forall v \in V, v+v=0$.

Proposition: Let $(V,+,\cdot)$ be a finite $\mathbb{Z}/2\mathbb{Z}$-vector space. Then there exists $n \geq 0$ such that V has cardinal 2^n, and $(V,+,\cdot)$ is isomorphic to the vector space $(\mathbb{Z}/2\mathbb{Z})^n$.

Proof: Consequence of the theory of vector spaces.

A **linear subspace** of $(V, +, \cdot)$ is a subset $W \subset V$ such that

$$\forall u, v \in W, u + v \in W \qquad \text{and} \qquad \forall v \in W, \forall \lambda \in \mathbb{Z}/2\mathbb{Z}, \lambda v \in W.$$

We define the following equivalence relation on V: for all $u, v \in V$,

$$u \sim v \iff u - v \in W.$$

Denote by V/W the quotient set of V under this relation. For any $v \in V$, one shows that the equivalence class of v is equal to $v + W = \{v + w \mid w \in W\}$.

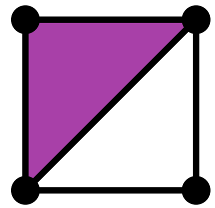
One defines a group structure $\oplus$ on V/W as follows:

$$(u + W) \oplus (u' + W) = (u + u') + W.$$

Definition: The vector space $(V/W, \oplus, \cdot)$ is called the **quotient vector space**.

Proposition: We have $\dim V/W = \dim V - \dim W$.

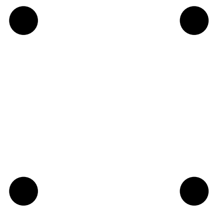
Definition (reminder): Let V be a set (called the set of *vertices*). A **simplicial complex** over V is a set K of subsets of V (called the *simplices*) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$.
The dimension of a simplex $\sigma \in K$ is $\dim(\sigma) = |\sigma| - 1$.



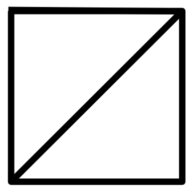
K

Let K be a simplicial complex. For any $n \geq 0$, define

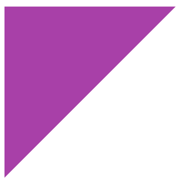
$$K_{(n)} = \{\sigma \in K \mid \dim(\sigma) = n\}.$$



$K_{(0)}$



$K_{(1)}$



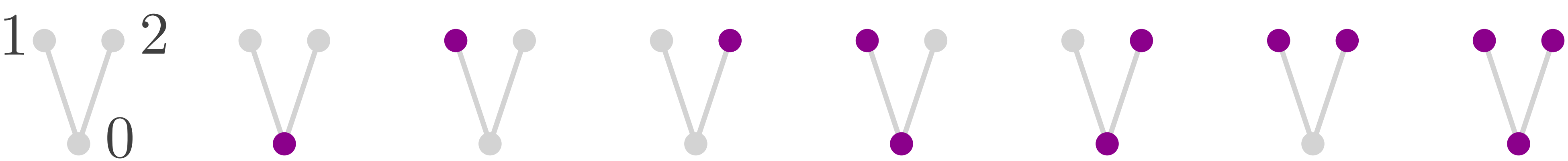
$K_{(2)}$

Cadeias

Let $n \geq 0$. The n-**chains** of K is the set $C_n(K)$ whose elements are the formal sums

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \quad \text{where} \quad \forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}.$$

Example: The 0-chains of $K = \{[0], [1], [2], [0,1], [0,2]\}$ are:



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and the 1-chains

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We can give $C_n(K)$ a **group structure** via

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma + \sum_{\sigma \in K_{(n)}} \eta_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} (\epsilon_\sigma + \eta_\sigma) \cdot \sigma.$$

Moreover, $C_n(K)$ can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure.

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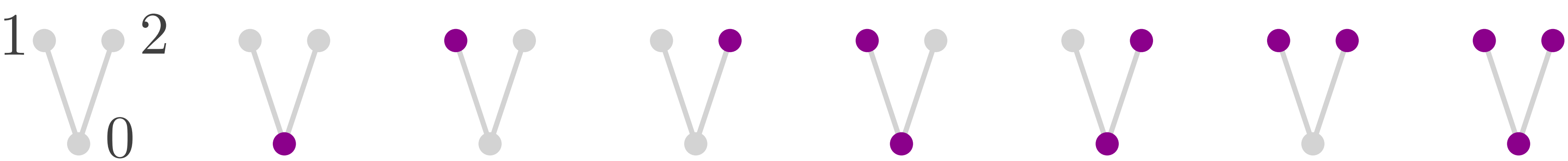

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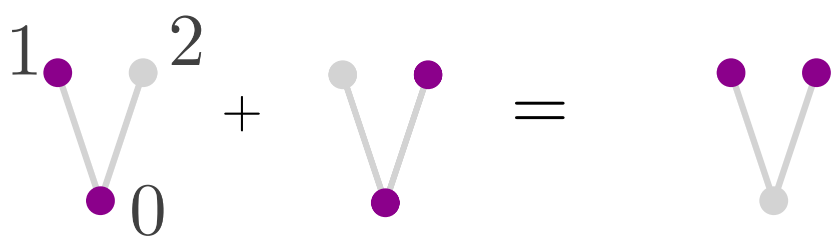
Moreover, $C_n(K)$ can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure.

Example: In the simplicial complex $K = \{[0], [1], [2], [0,1], [0,2]\}$, the sum of the 0-chains $[0] + [1]$ and $[0] + [2]$ is $[1] + [2]$:

$$([0] + [1]) + ([0] + [2]) = [0] + [0] + [1] + [2] = [1] + [2].$$



Operador bordo

Let $n \geq 1$, and $\sigma = [x_0, ..., x_n] \in K_{(n)}$ a simplex of dimension n. We define its **boundary** as the following element of $C_{n-1}(K)$:

$$\partial_n \sigma = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau$$

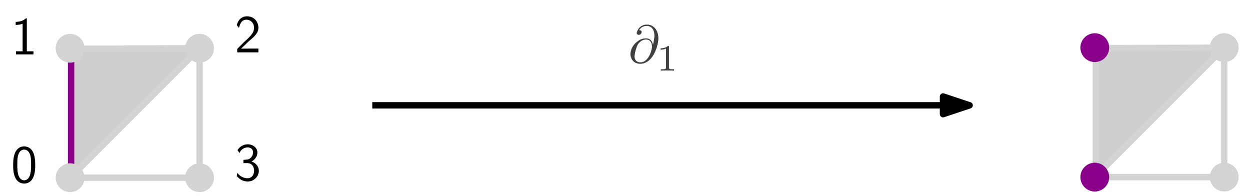
We can extend the operator ∂_n as a linear map $\partial_n \colon C_n(K) \to C_{n-1}(K)$.

Example: Consider the simplicial complex

$$K = \{[0], [1], [2], [3], [0,1], [0,2], [1,2], [1,3], [2,3], [0,1,2]\}.$$

The simplex $[0,1]$ has the faces $[0]$ and $[1]$. Hence

$$\partial_1 [0,1] = [0] + [1].$$



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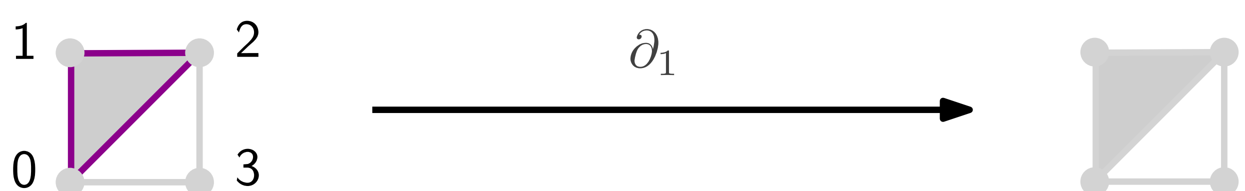
The boundary of the 1-chain $[0,1] + [1,2] + [2,0]$ is

$$\begin{aligned} \partial_1\big([0,1] + [1,2] + [2,0]\big) &= \partial_1[0,1] + \partial_1[1,2] + \partial_1[2,0] \\ &= [0] + [1] + [1] + [2] + [2] + [0] = 0 \end{aligned}$$



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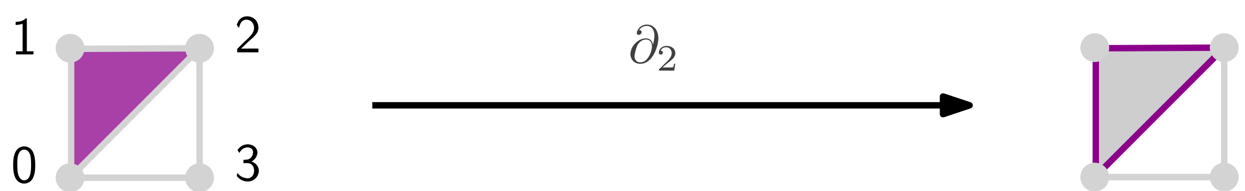
The simplex $[0,1,2]$ has the faces $[0,1]$ and $[1,2]$ and $[2,0]$. Hence

$$\partial_2 [0,1,2] = [0,1] + [1,2] + [2,0].$$



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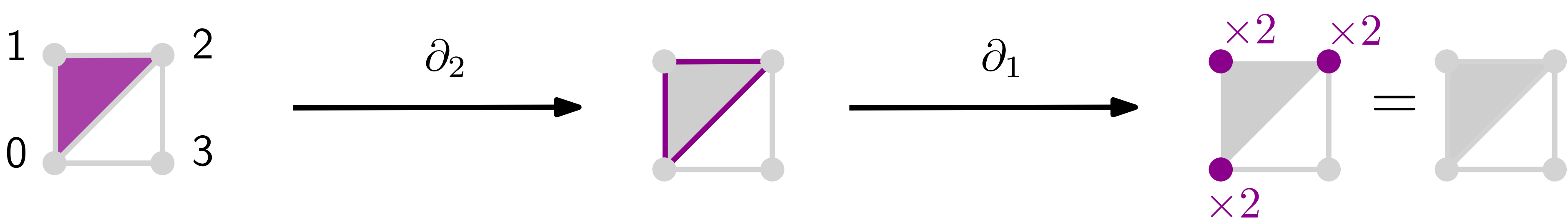
We can extend the operator ∂_n as a linear map $\partial_n : C_n(K) \to C_{n-1}(K)$.

Proposition: For any $n \geq 1$, for any $c \in C_n(K)$, we have $\partial_{n-1} \circ \partial_n(c) = 0$.



Proposition: For any $n \geq 1$, for any $c \in C_n(K)$, we have $\partial_{n-1} \circ \partial_n(c) = 0$.

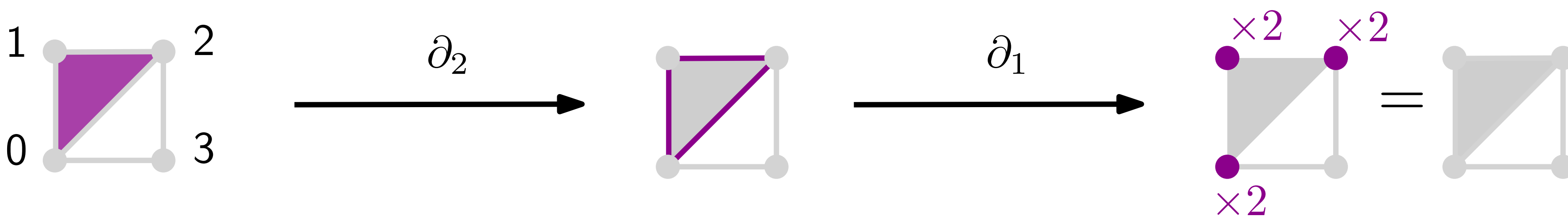
Proof: Suppose that $n \geq 2$, the result being trivial otherwise.
Since the boundary operators are linear, it is enough to prove that $\partial_{n-1} \circ \partial_n(\sigma) = 0$ for all simplex $\sigma \in K_{(n)}$.
By definition, $\partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau$, and

$$\partial_{n-1} \circ \partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \partial_{n-1}(\tau) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \sum_{\substack{\nu \subset \tau \\ |\nu| = |\tau| - 1}} \nu$$

We can write this last sum as

$$\sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \sum_{\substack{\nu \subset \tau \\ |\nu| = |\tau| - 1}} \nu = \sum_{\substack{\nu \subset \sigma \\ |\nu| = |\sigma| - 2}} \alpha_\nu \nu$$

where $\alpha_\nu = \{\tau \subset \sigma \mid |\tau| = |\sigma| - 1, \nu \subset \tau\}$.
It is easy to see that for every ν such that $\dim \nu = \dim \tau - 2$, we have $\alpha_\nu = 2 = 0$.



Ciclos e bordos

Let $n \geq 0$. We have a sequence of vector spaces

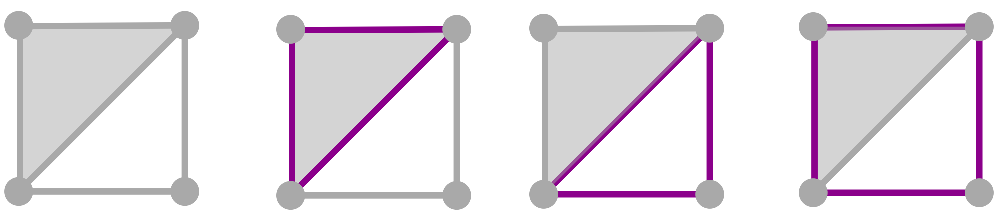
$$\cdots \longrightarrow C_{n+1}(K) \xrightarrow{\partial n+1} C_n(K) \xrightarrow{\partial n} C_{n-1}(K) \longrightarrow \ldots$$

The maps ∂_{n+1} and ∂_n are linear maps, and we can consider their kernel and image.

Definition: We define:

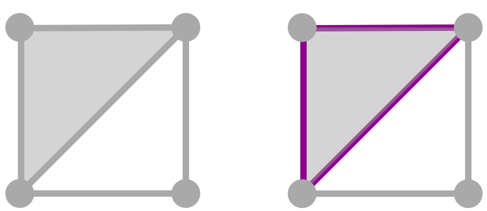
- The n-**cycles**: $Z_n(K) = \mathrm{Ker}(\partial_n) = \{c \in C_n(K) \mid \partial_n(c) = 0\}$,
- The n-**boundaries**: $B_n(K) = \mathrm{Im}(\partial_{n+1}) = \{\partial_{n+1}(c) \mid c \in C_{n+1}(K)\}$.

Example: The 1-cycles are:



0 $\quad$ $[0,1]+[1,2]+[0,2]$ $\quad$ $[0,2]+[2,3]+[0,3]$ $\quad$ $[0,1]+[1,2]+[2,3]+[0,3]$.

The 1-boundaries are:



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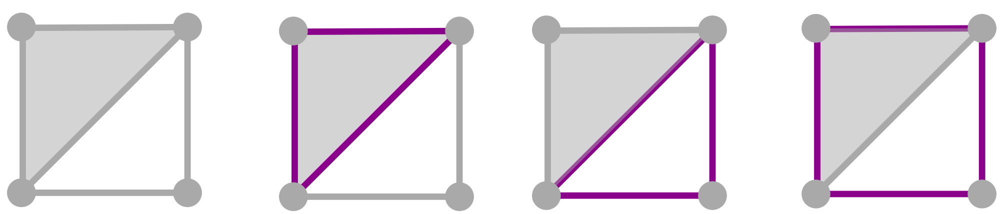
Example: The 1-cycles are:



0 $\quad$ $[0,1]+[1,2]+[0,2]$ $\quad$ $[0,2]+[2,3]+[0,3]$ $\quad$ $[0,1]+[1,2]+[2,3]+[0,3].$

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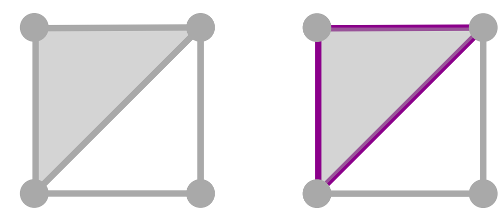


0 $\quad$ $[0,1]+[1,2]+[0,2]$

Proposition: We have $B_n(K) \subset Z_n(K)$.

$\longrightarrow$ interpretation: among the cycles, the boundaries are not actual ‘holes’ (they are filled)

Grupos de homologia

We have defined a sequence of vector spaces, connected by linear maps

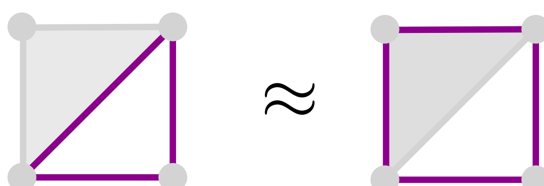
$$\cdots \longrightarrow C_{n+1}(K) \longrightarrow C_n(K) \longrightarrow C_{n-1}(K) \longrightarrow \cdots$$

and for every $n \geq 0$, we have defined the cycles and the boundaries $Z_n(K)$ and $B_n(K)$.

Definition: The n**th** **(simplicial) homology group** of K is the quotient vector space

$$H_n(K) = Z_n(K)/B_n(K).$$

⟶ interpretation: by taking the quotient, we kill the boundaries, and we are left only with the actual 'holes'



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Remark: A finite $\mathbb{Z}/2\mathbb{Z}$-vector space must be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some k.

Definition: Let K be a simplicial complex and $n \geq 0$. Its n^{th} **Betti number** is the integer $\beta_n(K) = \dim H_n(K)$.

$$H_n(K) = (\mathbb{Z}/2\mathbb{Z})^k \quad \longrightarrow \quad \beta_n(K) = k$$

Grupos de homologia

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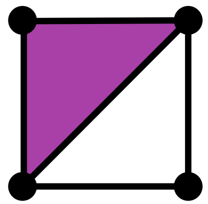
Example:

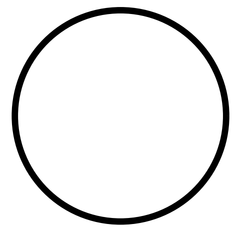
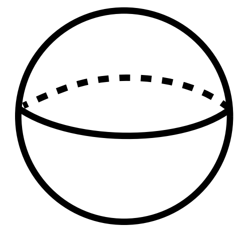
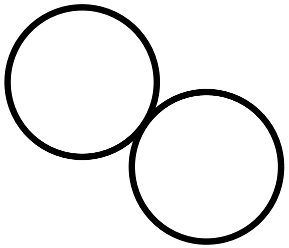
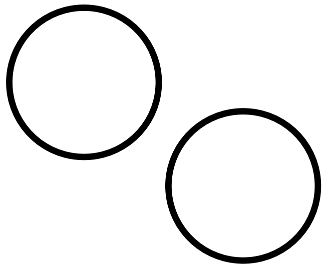


$H_0(K) = \mathbb{Z}/2\mathbb{Z} \longrightarrow \beta_0(K) = 1$

$H_1(K) = \mathbb{Z}/2\mathbb{Z} \longrightarrow \beta_1(K) = 1$

$H_2(K) = 0 \longrightarrow \beta_2(K) = 0$

Grupos de homologia

X					
$H_0(X)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\beta_0(X)$	1	1	1	1	2
$H_1(X)$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\beta_1(X)$	0	1	0	2	2
$H_2(X)$	0	0	$\mathbb{Z}/2\mathbb{Z}$	0	0
$\beta_2(X)$	0	0	1	0	0

Ordenar o complexo simplicial

Let K be a simplicial complex with n simplices. Choose a total order of the simplices

$$\sigma^1 < \sigma^2 < \ldots < \sigma^n$$

such that

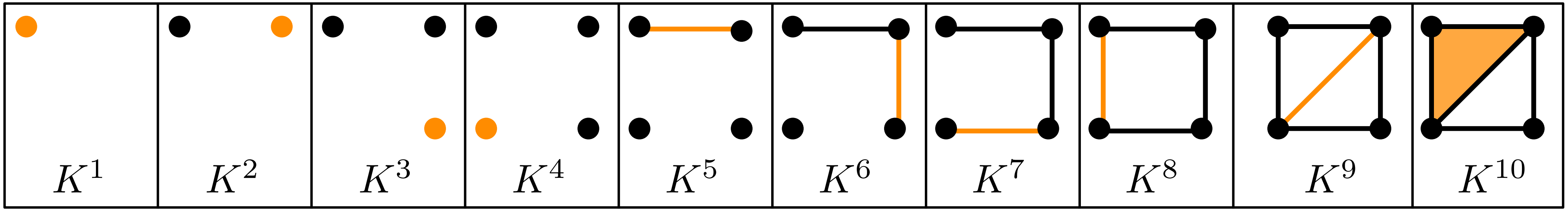
$$\forall \sigma, \tau \in K,\ \tau \subsetneq \sigma \implies \tau < \sigma.$$

In other words, a face of a simplex comes before the simplex itself.
For every $i \leq n$, consider the simplicial complex

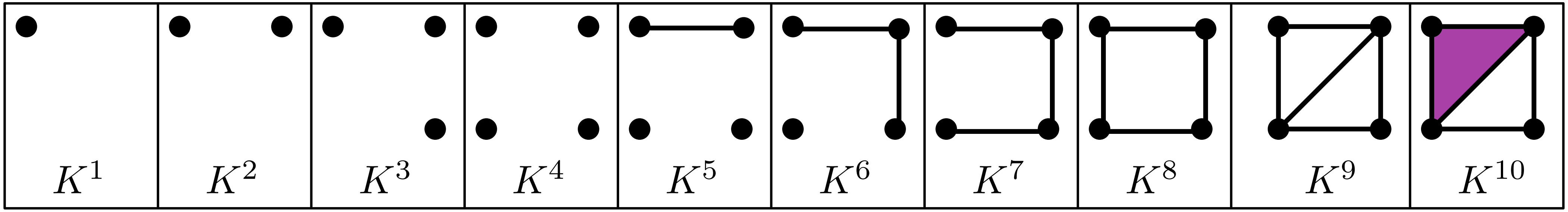
$$K^i = \{\sigma^1, \ldots, \sigma^i\}.$$

We have $\forall i \leq n, K^{i+1} = K^i \cup \left\{\sigma^{i+1}\right\}$, and $K^n = K$. They form an inscreasing sequence of simplicial complexes

$$K^1 \subset K^2 \subset \ldots \subset K^n.$$



Positividade dos simplexos



Let $k \geq 0$. We will compute the homology groups of K^i incrementally:

$H_k(K^1),\ H_k(K^2),\ H_k(K^3),\ H_k(K^4),\ H_k(K^5),\ H_k(K^6),\ H_k(K^7),\ H_k(K^8),\ H_k(K^9),\ H_k(K^{10})$

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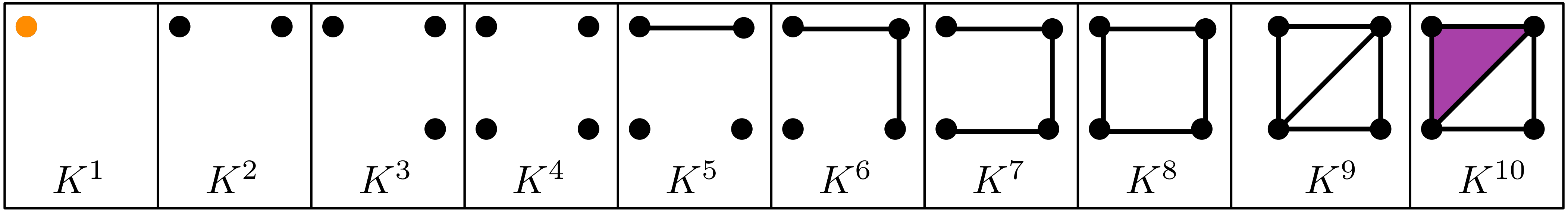

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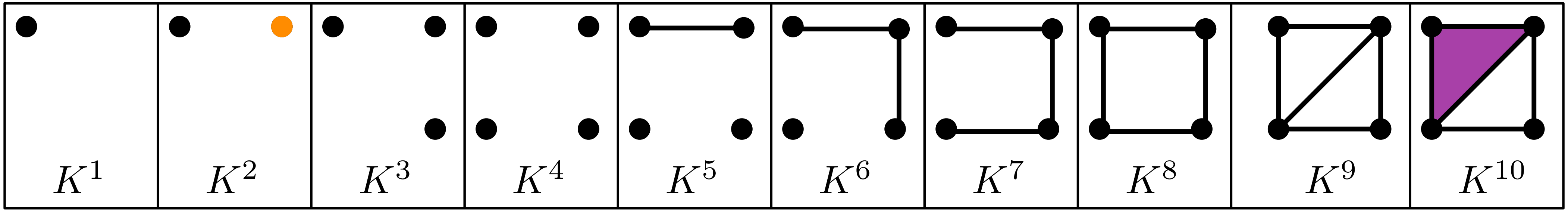
Definition: Let $i \in [\![1, n]\!]$, and $d = \dim(\sigma^i)$. Recall that $K^i = K^{i-1} \cup \{\sigma_i\}$.
The simplex σ^i is **positive** if there exists a cycle $c \in Z_d(K^i)$ that contains σ^i.
In other words, there exist $c = \sum_{\sigma \in K^i_{(n)}} \epsilon_\sigma \cdot \sigma \in C_n(K^i)$ such that $\epsilon_{\sigma^i} = 1$ and $\partial_n(c) = 0$. Otherwise, σ^i is **negative**.

Example:

- $\sigma^1 \in K^1$ is **positive** because it is included in the cycle $c = \sigma^1$ (indeed, $\partial_0(\sigma^1) = 0$).

Positividade dos simplexos



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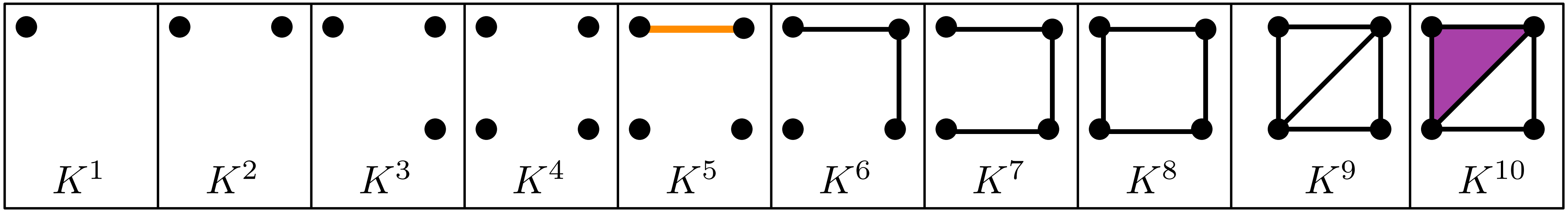
Positividade dos simplexos



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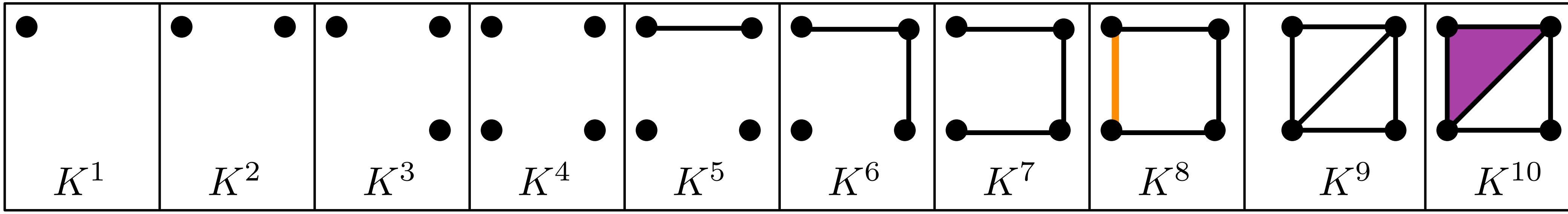
- $\sigma^6 \in K^5$ is **negative** because it is not included in a cycle $Z_1(K^5)$. Indeed, $C_1(K^5)$ only contains 0 and σ_5, and $\partial_1(\sigma^5) = \sigma^1 + \sigma^2 \neq 0$.

Positividade dos simplexos



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- $\sigma^8 \in K^8$ is **positive** because it is included in the cycle $c = \sigma^5 + \sigma^6 + \sigma^7 + \sigma^8$ (indeed, $\partial_1(c) = 2\sigma^1 + 2\sigma^2 + 2\sigma^3 + 2\sigma^4 = 0$).

Positividade dos simplexos

Definition: Let $i \in [\![1, n]\!]$, and $d = \dim(\sigma_i)$. Recall that $K^i = K^{i-1} \cup \{\sigma_i\}$.
The simplex σ_i is **positive** if there exists a cycle $c \in Z_d(K^i)$ that contains σ_i.
Otherwise, σ_i is **negative**.

Remark: By adding σ^i in the simplicial complex, the only groups that may change are $Z_d(K^i)$ and $B_{d-1}(K^i)$.

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Lemma: If σ^i is positive, then $\beta_d(K^i) = \beta_d(K^{i-1}) + 1$,
and for all $d' \neq d$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.

Proof: We start by proving the following fact: if $c \in Z_d(K^i)$ is a cycle that contains σ_i, then c is not homologous (in K^i) to a cycle of $c' \in Z_d(K^{i-1})$.

By contradiction: if $c = c' + b$ with $c' \in Z_d(K^{i-1})$ and $b \in B_d(K^i)$, then $c - c' = b \in B_d(K^i)$. This is absurd because we just added σ_i: it cannot appear in a boundary of K^i.

As a consequence, $\dim Z_d(K^i) = \dim Z_d(K^{i-1}) + 1$.

We conclude by using the relation $\beta_d(K^i) = \dim Z_d(K^i) - \dim B_d(K^i)$.

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Lemma: If σ^i is negative, then $\beta_{d-1}(K^i) = \beta_{d-1}(K^{i-1}) - 1$,
and for all $d' \neq d-1$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.

Proof: We start by proving the following fact: $\partial_d(\sigma^i)$ is not a boundary of K^{i-1}.

Otherwise, we would have $\partial_d(\sigma^i) = \partial_d(c)$ with $c \in C_d(K^{i-1})$, i.e. $\partial_d(\sigma^i + c) = 0$.
Hence $\sigma^i + c$ would be a cycle of K^i that contains c, contradicting the negativity of σ^i.

As a consequence, $\dim B_{d-1}(K^i) = \dim B_{d-1}(K^{i-1}) + 1$.

We conclude by using the relation $\beta_{d-1}(K^i) = \dim Z_{d-1}(K^i) - \dim B_{d-1}(K^i)$.

Algoritmo incremental

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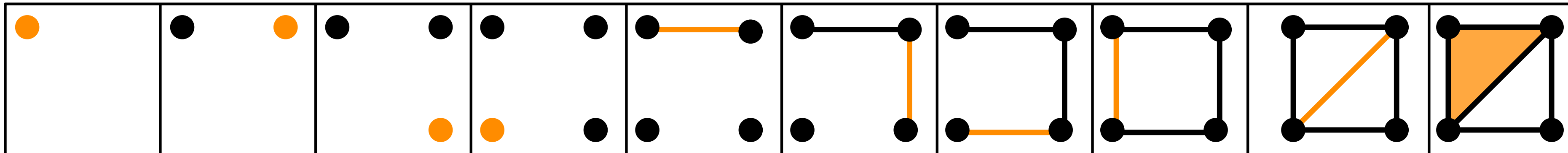
We deduce the following algorithm:

Input: an increasing sequence of simplicial complexes $K^1 \subset \cdots \subset K^n = K$
Output: the Betti numbers $\beta_0(K), ...\beta_d(K)$
$\beta_0 \leftarrow 0, ..., \beta_d \leftarrow 0$;
for $i \leftarrow 1$ **to** n **do**
    $d = \dim(\sigma^i)$;
    if σ^i *is positive* **then**
        $\beta_k(K^i) \leftarrow \beta_k(K^i) + 1$;
    else if $d > 0$ **then**
        $\beta_{k-1}(K^i) \leftarrow \beta_{k-1}(K^{i-1}) - 1$;

Algoritmo incremental



	K^1	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}
Dimension	0	0	0	0	1	1	1	1	1	2
Positivity	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$
$\beta_0(K^i)$	1	2	3	4	3	2	1	1	1	1
$\beta_1(K^i)$	0	0	0	0	0	0	0	1	2	1

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  $d = \dim(\sigma^i)$;
  if σ^i *is positive* **then**
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  else if $d > 0$ **then**
    $\beta_{k-1}(K^i) \leftarrow \beta_{k-1}(K^{i-1}) - 1$;

Característica de Euler

Reminder: the Euler characteristic of a simplicial complex K is

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Proposition: The Euler characteristic is also equal to

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot \beta_i(K).$$

Característica de Euler

Proposition: The Euler characteristic of K is equal to

$$\chi(K) = \sum_{0 \le i \le n} (-1)^i \cdot \beta_i(K).$$

Proof: Pick an ordering $K^1 \subset \cdots \subset K^n = K$ of K, with $K^i = K^{i-1} \cup \{\sigma^i\}$ for all $2 \le i \le n$.

By induction, let us show that, for all $1 \le m \le n$,

$$\sum_{0 \le i \le m} (-1)^i \cdot \beta_i(K^m) = \sum_{0 \le i \le m} (-1)^i \cdot (\text{number of simplices of dimension } i \text{ of } K^m).$$

For $m = 1$, σ^m is a 0-simplex, and the equality reads $1 = 1$.

Now, suppose that the equality is true for $1 \le m < n$, and consider the simplex σ^{m+1}. Let $d = \dim \sigma^{m+1}$. The right-hand side of the Equation is increased by $(-1)^d$.

If σ^{m+1} is positive, then $\beta_d(K^{m+1}) = \beta_d(K^m) + 1$, hence the left-hand side of the Equation is increased by $(-1)^d$.

Otherwise, it is negative, and $\beta_{d-1}(K^{m+1}) = \beta_{d-1}(K^m) - 1$, hence the left-hand side of the Equation is increased by $-(-1)^{d-1} = (-1)^d$.

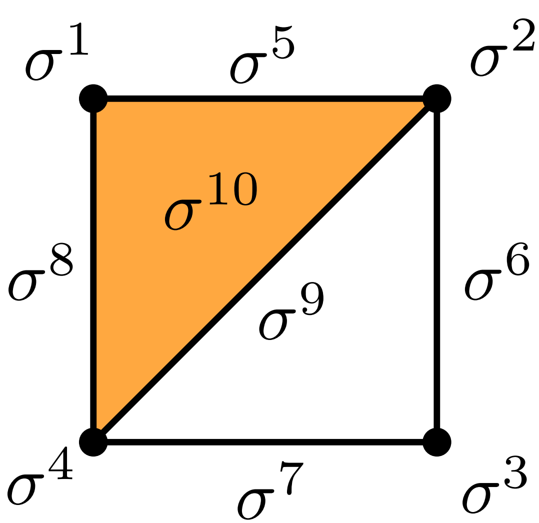
Matriz de bordo

The only thing missing to apply the incremental algorithm is to determine whether a simplex is positive or negative.

Let K be a simplicial complex, and $\sigma^1 < \sigma^2 < \cdots < \sigma^n$ and ordering of its simplices.

Define the **boundary matrix** of K, denoted Δ, as follows: Δ is a $n \times n$ matrix, whose (i,j)-entry (i^{th} row, j^{th} column is)

$$\begin{aligned} \Delta_{i,j} &= 1 \text{ if } \sigma^i \text{ is a face of } \sigma^j \text{ and } |\sigma^i| = |\sigma^j| - 1 \\ & \quad 0 \text{ else.} \end{aligned}$$



$$\begin{array}{c} \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Matriz de bordo

By adding columns one to the others, we create chains.
If we were able to reduce a column to zero, then we found a cycle.

$$
\begin{array}{c} \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$\partial_1(\sigma^6) = \sigma^2 + \sigma^3$$

$$
\begin{array}{c} \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$\partial_1(\sigma^5 + \sigma^6 + \sigma^7 + \sigma^8) = 0$$

Matriz de bordo

The process of reducing columns to zero is called **Gauss reduction**.
For any $j \in [\![1, n]\!]$, define
$$\delta(j) = \max\{i \in [\![1, n]\!] \mid \Delta_{i,j} \neq 0\}.$$

If $\Delta_{i,j} = 0$ for all j, then $\delta(j)$ is *undefined*.

We say that the boundary matrix Δ is *reduced* if the map δ is injective on its domain of definition.

$$\begin{array}{c|cccccccccc|}
 & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \\
\sigma^1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\sigma^2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\sigma^3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\sigma^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\sigma^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Matriz de bordo

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

$$
\begin{array}{c}
\begin{array}{cccccccccc}
\sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10}
\end{array} \\
\begin{array}{c}
\sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10}
\end{array}
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccccccc}
\sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8+\sigma^7 & \sigma^9 & \sigma^{10}
\end{array} \\
\begin{array}{c}
\sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10}
\end{array}
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Matriz de bordo

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Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
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    add column i to column j;

$$
\begin{array}{c|cccccccccc}
 & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8+\sigma^7 & \sigma^9 & \sigma^{10} \\
\hline
\sigma^1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\sigma^2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
\sigma^3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\sigma^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\sigma^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{c|cccccccccc}
 & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8+\sigma^7+\sigma^6 & \sigma^9 & \sigma^{10} \\
\hline
\sigma^1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\sigma^2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\sigma^3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\sigma^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\sigma^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Matriz de bordo

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ

Output: a reduced matrix $\widetilde{\Delta}$

for $j \leftarrow 1$ **to** n **do**

  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**

    add column i to column j;

$$
\begin{array}{c}
\\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
\sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8+\sigma^7+\sigma^6 & \sigma^9 & \sigma^{10}
\end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{c}
\\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
\sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8+\sigma^7+\sigma^6+\sigma^5 & \sigma^9 & \sigma^{10}
\end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Matriz de bordo

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ

Output: a reduced matrix $\widetilde{\Delta}$

for $j \leftarrow 1$ **to** n **do**

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    add column i to column j;

$$
\begin{array}{c|cccccccccc}
 & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8+\sigma^7+\sigma^6+\sigma^5 & \sigma^9 & \sigma^{10} \\
\hline
\sigma^1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\sigma^2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
\sigma^3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\sigma^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\sigma^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{c|cccccccccc}
 & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \\
\hline
\sigma^1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\sigma^2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\sigma^3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\sigma^4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\sigma^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\sigma^{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Matriz de bordo

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

Lemma: Suppose that the boundary matrix is reduced. Let $j \in [\![1, n]\!]$.
If $\delta(j)$ is defined, then the simplex σ^j is negative.
Otherwise, it is positive.

$$
\begin{array}{cc}
 & \begin{matrix} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \end{matrix} \\
\begin{matrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{matrix} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
 & \begin{matrix} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \\ + & + & + & + & - & - & - & + & + & - \end{matrix}
\end{array}
$$

Algoritmo final

Incremental computation of the homology

Input: an increasing sequence of simplicial complexes $K^1 \subset \cdots \subset K^n = K$
Output: the Betti numbers $\beta_0(K), ...\beta_d(K)$
$\beta_0 \leftarrow 0, ..., \beta_d \leftarrow 0$;
for $i \leftarrow 1$ **to** n **do**
  $d = \dim(\sigma^i)$;
  if σ^i *is positive* **then**
    $\beta_k(K^i) \leftarrow \beta_k(K^i) + 1$;
  else if $d > 0$ **then**
    $\beta_{k-1}(K^i) \leftarrow \beta_{k-1}(K^{i-1}) - 1$;

Gauss reduction of the boundary matrix

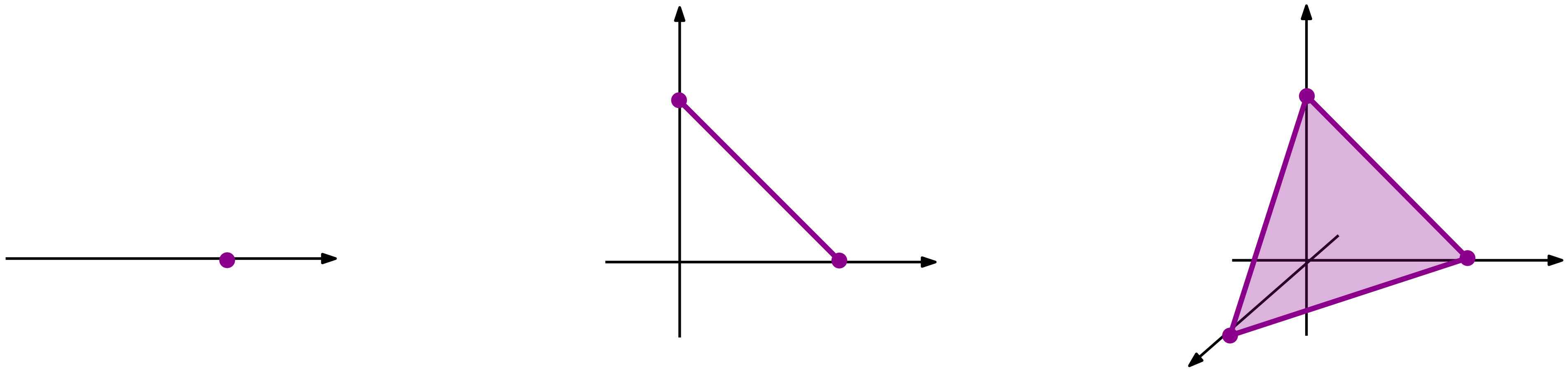
Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $i \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

Simplexo padrão

In order to describe topological spaces, we will decompose them into simpler pieces. The pieces we shall consider are the standard simplices.

The **standard simplex of dimension** n is the following subset of $\mathbb{R}^{n+1}$

$$\Delta_n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1, ..., x_{n+1} \geq 0 \text{ and } x_1 + ... + x_{n+1} = 1\}$$



Δ_0 Δ_1 Δ_2

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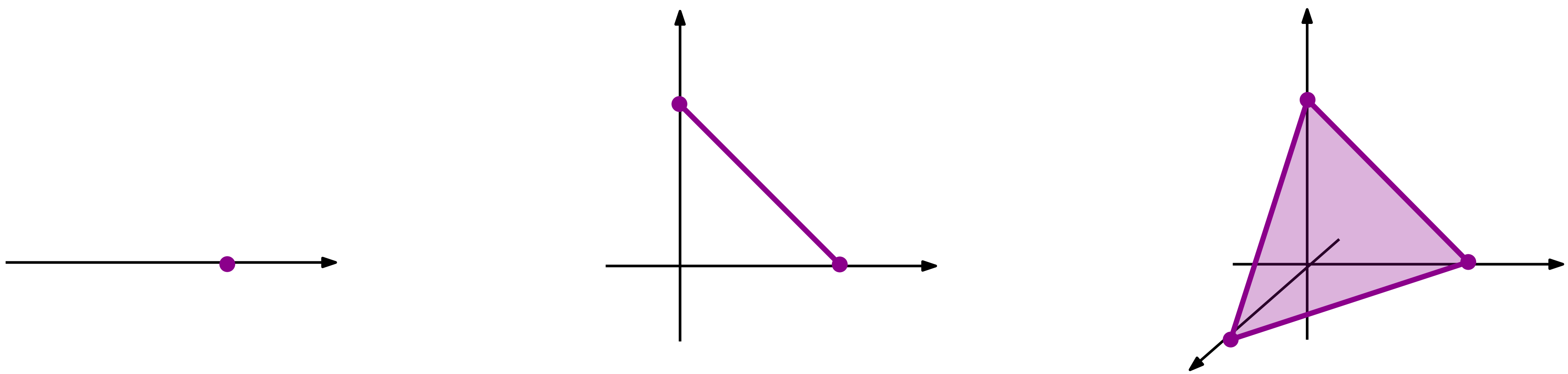

Δ_0 Δ_1 Δ_2

Remark: For any collection of points $a_1, ..., a_k \in \mathbb{R}^n$, their convex hull is defined as:

$$\text{conv}(\{a_1...a_k\}) = \left\{\sum_{1\leq i\leq k} t_i a_i \mid t_1 + ... + t_k = 1, \;\; t_1, ..., t_k \geq 0\right\}.$$

We can say that Δ_n is the convex hull of the vectors $e_1, ..., e_{n+1}$ of $\mathbb{R}^{n+1}$, where

$$e_i = (0, ..., 1, 0, ..., 0) \qquad (i^{\text{th}} \text{ coordinate } 1, \text{ the other ones } 0).$$

Realização topológica

Let us give simplicial complexes a topology.

Definition: Let K be a simplicial complex, with vertex $V = \{1, ..., n\}$.
In $\mathbb{R}^n$, consider, for every $i \in [\![1, n]\!]$, the vector $e_i = (0, ..., 1, 0, ..., 0)$ (i^{th} coordinate 1, the other ones 0).
Let $|K|$ be the subset of $\mathbb{R}^n$ defined as:

$$|K| = \bigcup_{\sigma \in K} \text{conv}\left(\{e_j, j \in \sigma\}\right)$$

where conv represent the convex hull of points.
Endowed with the subspace topology, $(|K|, \mathcal{T}_{||K|})$ is a topological space, that we call the *topological realization of* K.

If $a_1, ..., a_k \in \mathbb{R}^n$, the convex hull is defined as:

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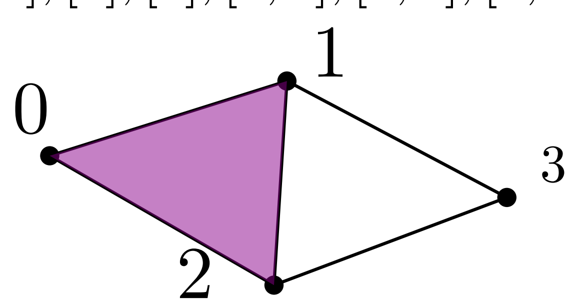
where conv represent the convex hull of points.
Endowed with the subspace topology, $(|K|, \mathcal{T}_{||K|})$ is a topological space, that we call the *topological realization of* K.

Remark: If the simplicial complex can be drawn in the plane (or space) without crossing itself, then its topological realization simply is the subspace topology.

Example: $K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 0], [1, 3], [2, 3], [0, 1, 2]\}$.

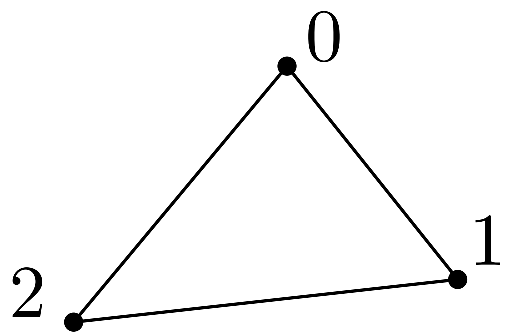


Triangulações

Definition: Let X be a topological space. A **triangulation** of X is a simplicial complex K such that its topological realization $|K|$ is homeomorphic to X.

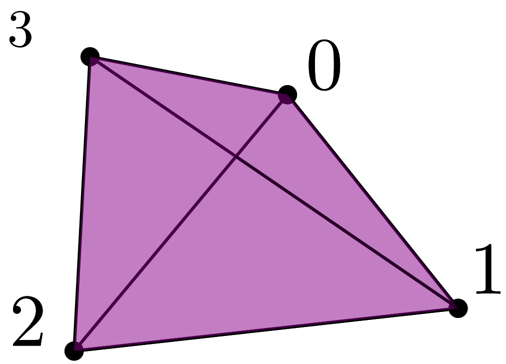
Example: The following simplicial complex is a triangulation of the circle:

$$K = \{[0], [1], [2], [0,1], [1,2], [2,0]\}$$



Example: The following simplicial complex is a triangulation of the sphere:

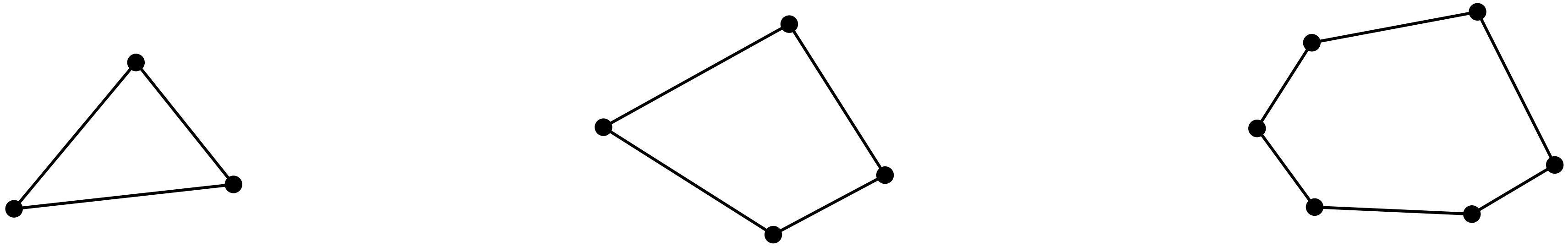
$$K = \{[0], [1], [2], [3], [0,1], [1,2], [2,3], [3,0], [0,2], [1,3], [0,1,2], [0,1,3], [0,2,3], [1,2,3]\}.$$



Triangulações

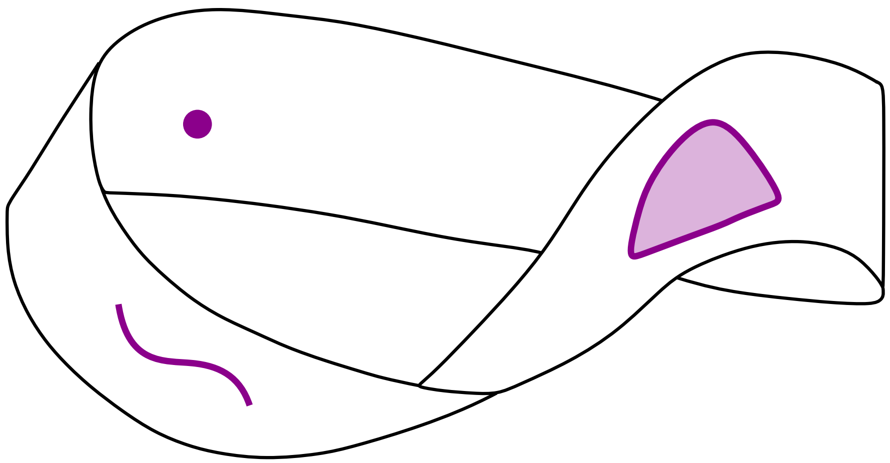
Definition: Let X be a topological space. A **triangulation** of X is a simplicial complex K such that its topological realization $|K|$ is homeomorphic to X.

Given a topological space, it is not always possible to triangulate it.
However, when it is, there exists many different triangulations.



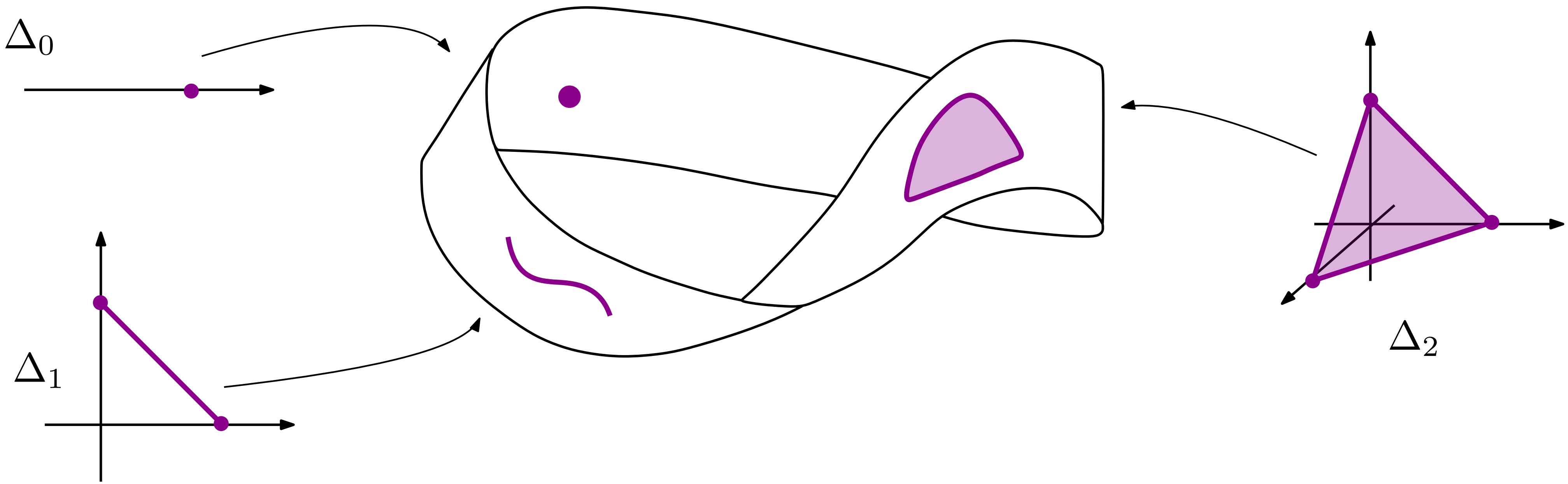
Theorem (Manolescu, 2016): For any dimension $n \geq 5$ there is a compact topological manifold which does not admit a triangulation.

Let us consider a **topological space** X. We want a notion of **simplices**.



Simplexo singular

Let us consider a **topological space** X. We want a notion of **simplices**.

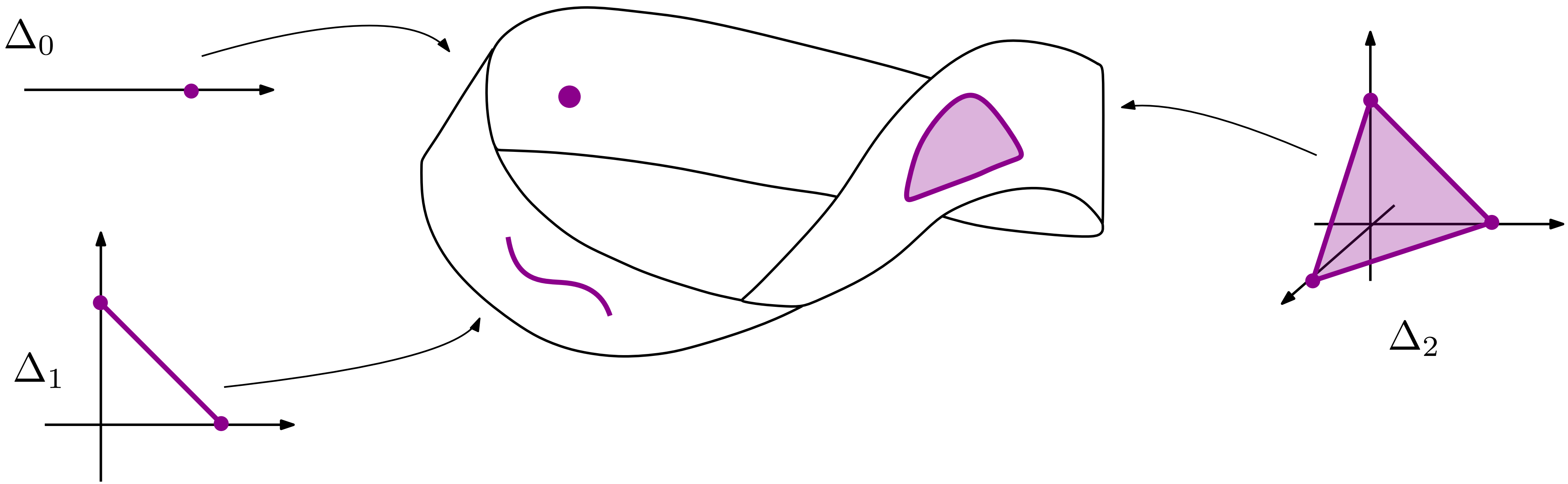


Definition: A **singular** n-**simplex** is a continuous map $\Delta_n \to X$, where Δ_n is the standard n-simplex. We denote S_n their set.

We now want a notion of **boundary**.

Simplexo singular

Let us consider a **topological space** X. We want a notion of **simplices**.



Definition: A **singular** n-**simplex** is a continuous map $\Delta_n \to X$, where Δ_n is the standard n-simplex. We denote S_n their set.

We now want a notion of **boundary**.

The boundary of Δ_n consists in $n+1$ copies of Δ_{n-1}.

We can restrict a singular n-simplex $\Delta_n \to X$ to the boundaries, giving $n+1$ singular $(n-1)$-simplices $\Delta_{n-1} \to X$.

Definition: The **boundary** of a singular n-simplex $\Delta_n \to X$ is the formal sum of the $n+1$ singular $(n-1)$-simplices $\Delta_{n-1} \to X$

For a **simplicial complex** K, we have defined

n-chains $\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma$ where $\forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}$

boundary operator $\partial_n \sigma = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau$

chain complex $\ldots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \ldots$

n-cycles and n-boundaries $Z_n(K) = \mathrm{Ker}(\partial_n)$ $B_n(K) = \mathrm{Im}(\partial_{n+1})$

n^{th}**simplicial** homology group $H_n(K) = Z_n(K)/B_n(K)$

Homologia singular

For a **simplicial complex** K, we have defined

n-chains $\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma$ where $\forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}$

boundary operator $\partial_n \sigma = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau$

chain complex $\dots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots$

n-cycles and n-boundaries $Z_n(K) = \mathrm{Ker}(\partial_n)$ $B_n(K) = \mathrm{Im}(\partial_{n+1})$

n^{th}**simplicial** homology group $H_n(K) = Z_n(K)/B_n(K)$

For a **topological space** X, we can define

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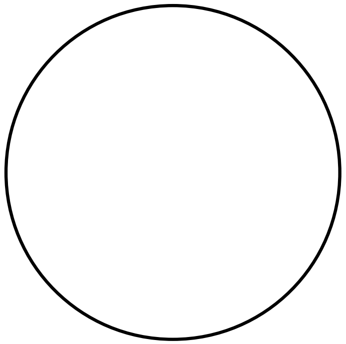
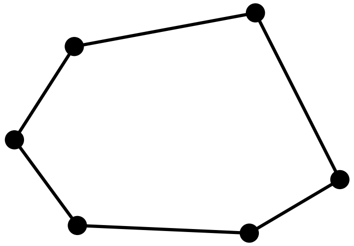
n^{th}**singular** homology group $H_n(X) = Z_n(X)/B_n(X)$

Homologia singular

Theorem: If X is a topological space and K a triangulation of it, then for all $n \geq 0$, $H_n(X) = H_n(K)$ (singular homology is equal to simplicial homology).





$H_0(X) = \mathbb{Z}/2\mathbb{Z}$

$H_1(X) = \mathbb{Z}/2\mathbb{Z}$

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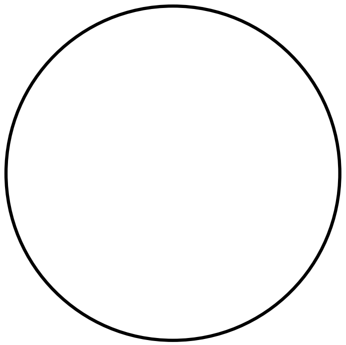

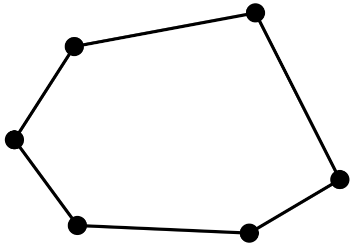

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$$H_0(K) = \mathbb{Z}/2\mathbb{Z}$$
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$$H_2(X) = 0$$

Theorem: If X and Y are homotopy equivalent topological spaces, then for all $n \geq 0$, $H_n(X) = H_n(Y)$.

Corollary: If K and L are homotopy equivalent simplicial complexes, then for all $n \geq 0$, $H_n(K) = H_n(L)$.

the homology groups are **invariants** of homotopy classes

Homologia é um functor

We have seen that homology transforms topological spaces into vector spaces

$$\begin{aligned} H_i \colon \mathrm{Top} &\longrightarrow \mathrm{Vect} \\ X &\longmapsto H_i(X) \end{aligned}$$

Actually, it also transforms **continous maps** into **linear maps**

$$X \xrightarrow{f} Y \qquad\qquad H_n(X) \xrightarrow{H_n(f)} H_n(Y)$$

This operation preserves **commutative diagrams**:

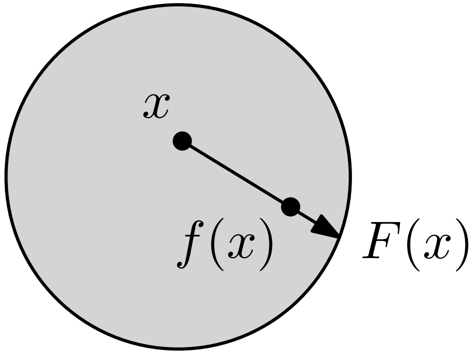
$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] & Y \arrow[r, "g"'] & Z, \end{tikzcd} \qquad \begin{tikzcd} H_n(X) \arrow[r, "H_n(f)"'] \arrow[rr, bend left, "H_n(g\circ f)"] & H_n(Y) \arrow[r, "H_n(g)"'] & H_n(Z). \end{tikzcd}$$

$$\boxed{H_n(g \circ f) = H_n(g) \circ H_n(f)}$$

Application (Brouwer's fixed point theorem):

Let $f\colon \mathcal{B} \to \mathcal{B}$ be a continous map, where $\mathcal{B}$ is the unit closed ball of $\mathbb{R}^n$. Let us show that f has a fixed point ($f(x) = x$).

If not, we can define a map $F\colon \mathcal{B} \to \partial\mathcal{B}$ such that F restricted to $\partial\mathcal{B}$ is the identity. To do so, define $F(x)$ as the first intersection between the half-line $[x, f(x))$ and $\partial\mathcal{B}$.



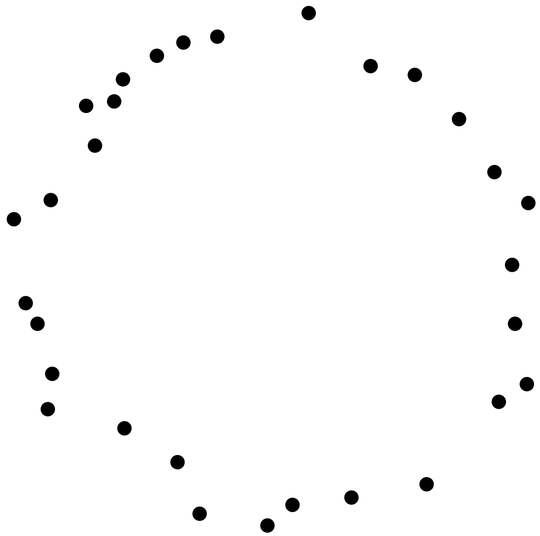
Denote the inclusion $i\colon \partial\mathcal{B} \to \mathcal{B}$. Then $F \circ i\colon \partial\mathcal{B} \to \partial\mathcal{B}$ is the identity.
By functoriality, we have commutative diagrams

$$\begin{tikzcd} \partial\mathcal{B} \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}"] & \mathcal{B} \arrow[r, "F"'] & \partial\mathcal{B}, \end{tikzcd} \qquad \begin{tikzcd} H_i(\partial\mathcal{B}) \arrow[r, "H_i(i)"'] \arrow[rr, bend left, "H_i(\mathrm{id})"] & H_i(\mathcal{B}) \arrow[r, "H_i(F)"'] & H_i(\partial\mathcal{B}). \end{tikzcd}$$

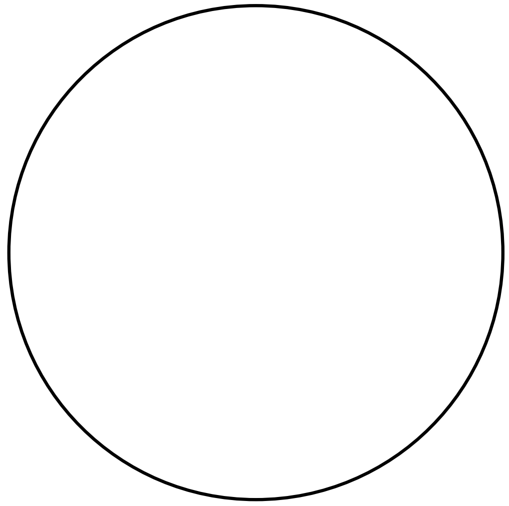
But for $i = n - 1$, we have an absurdity:

$$\begin{tikzcd} \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[rr, bend left, "\text{id}"] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z}. \end{tikzcd}$$

Let $\mathcal{M} \subset \mathbb{R}^n$ be a bounded subset.
Suppose that we are given a finite sample $X \subset \mathcal{M}$.
Estimate the homology groups of $\mathcal{M}$ from X.



X



$\mathcal{M}$

O problema da inferência homológica

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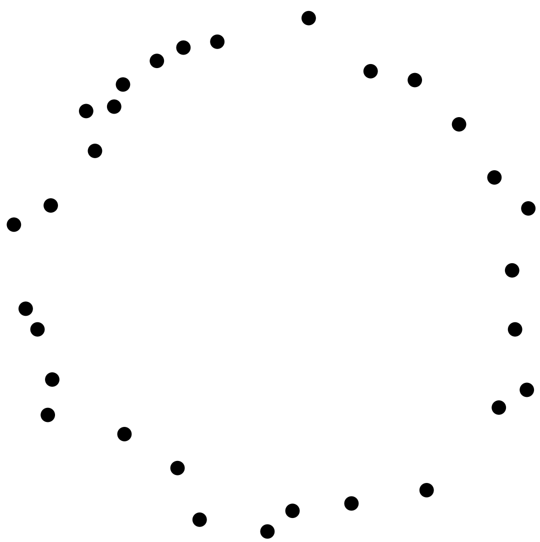

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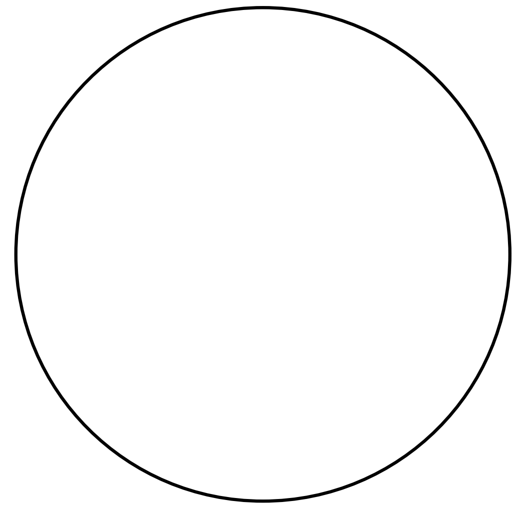

$\mathcal{M}$

We cannot use X directly. Its homology is disapointing:

$$\beta_0(X) = 30 \quad \text{and} \quad \beta_i(X) = 0 \text{ for } i \geq 1$$

number of connected components
$=$ number of points of X

O problema da inferência homológica

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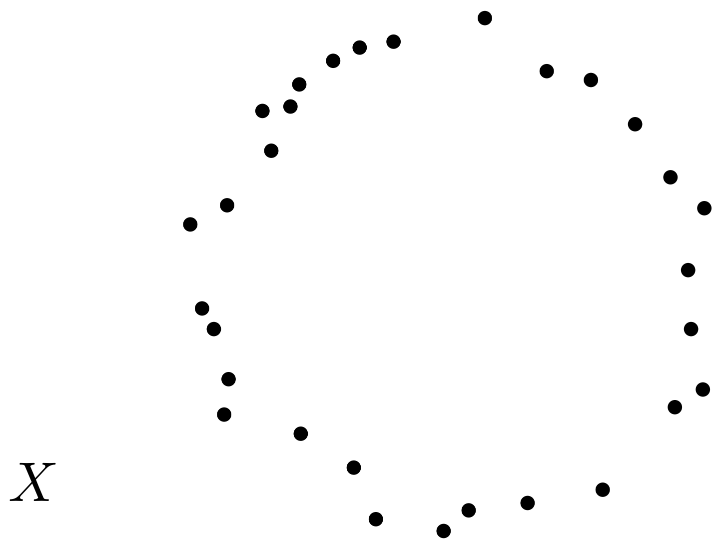

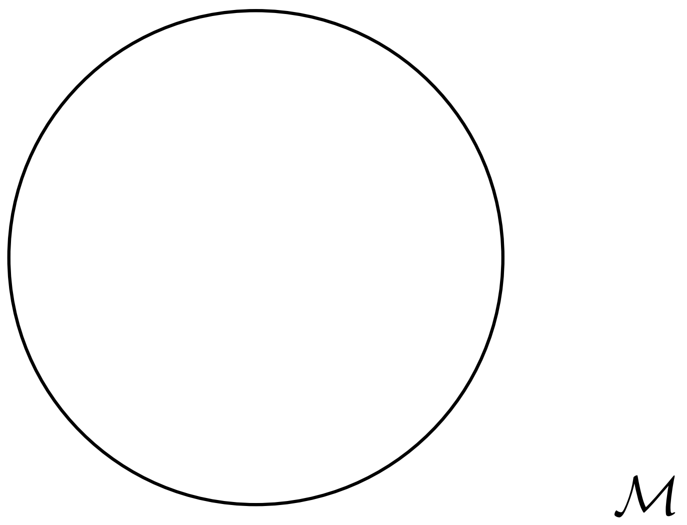

We cannot use X directly.

Idea: Thicken X.

Definition: For every $t \geq 0$, the t-**thickening** of the set X, denoted X^t, is the set of points of the ambient space with distance at most t from X:

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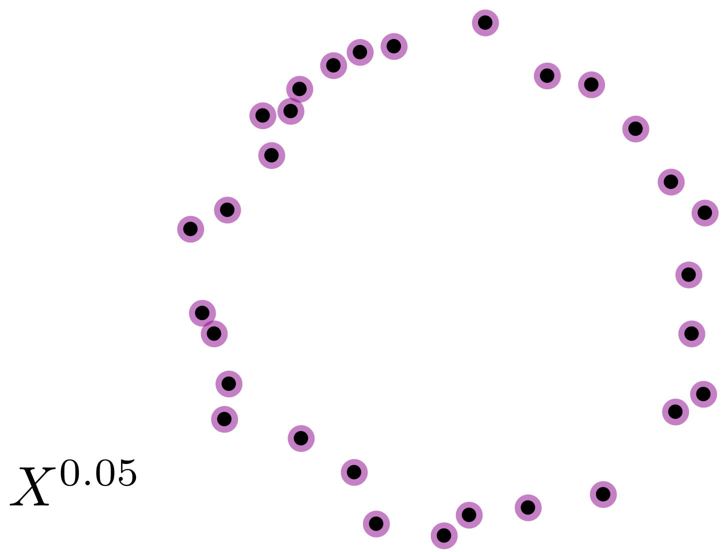

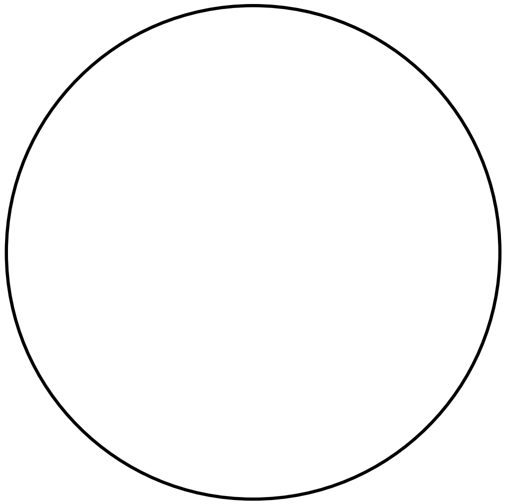

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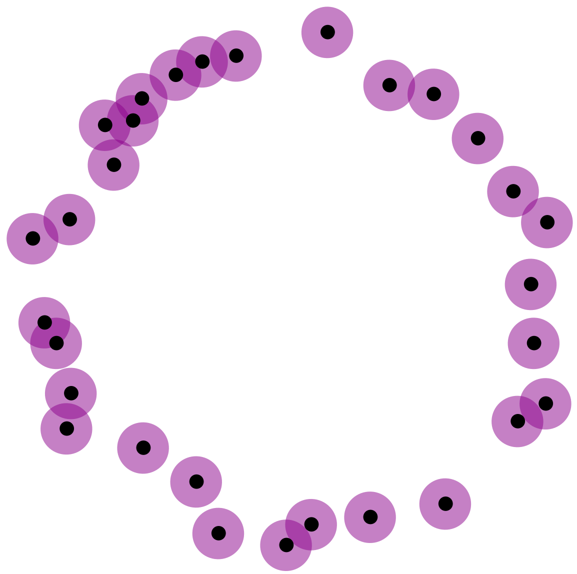
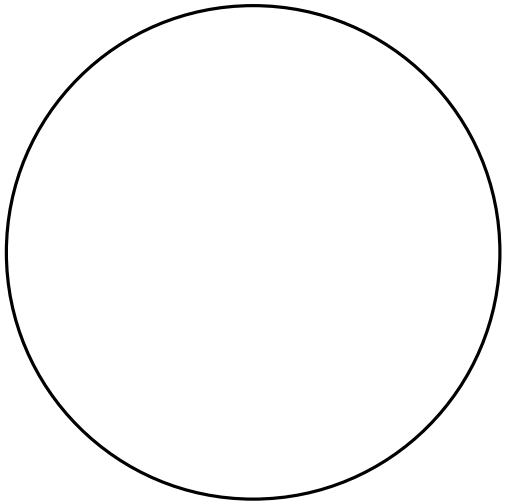
$X^{0.1}$





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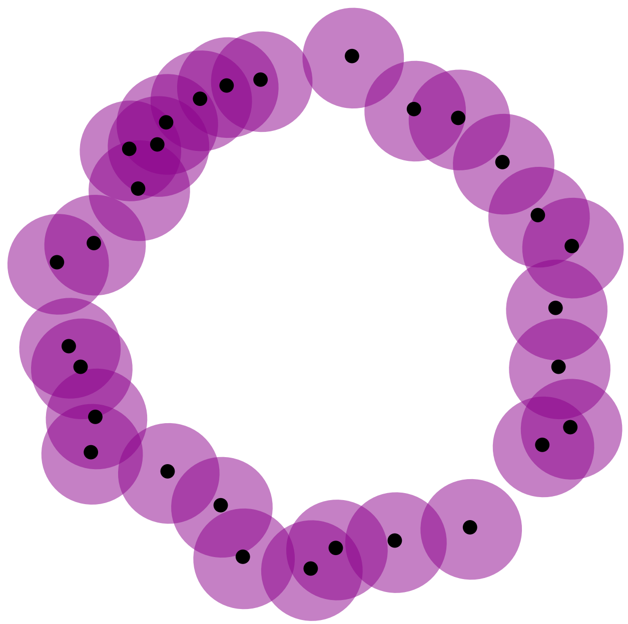
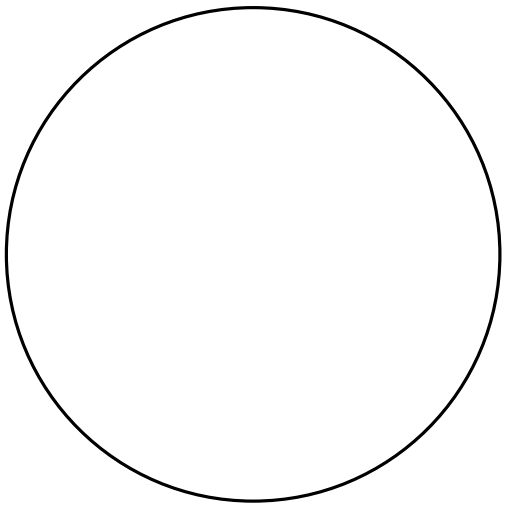


$X^{0.2}$



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O problema da inferência homológica

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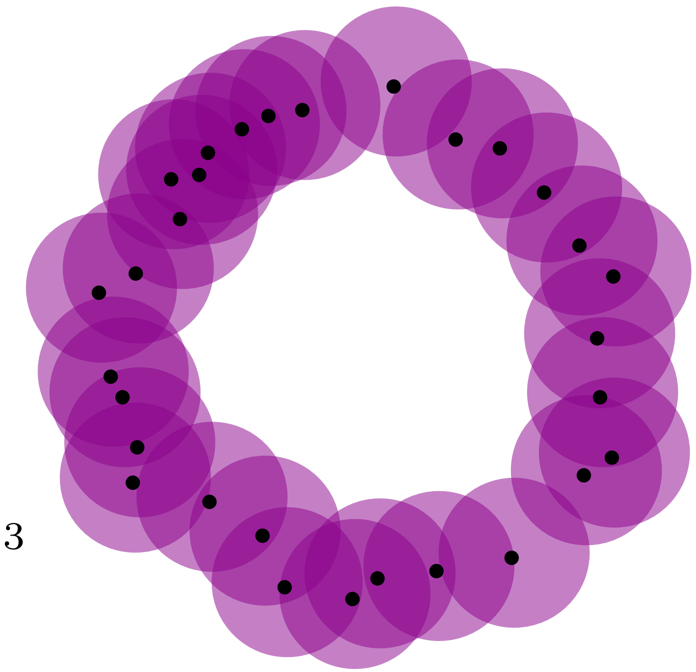
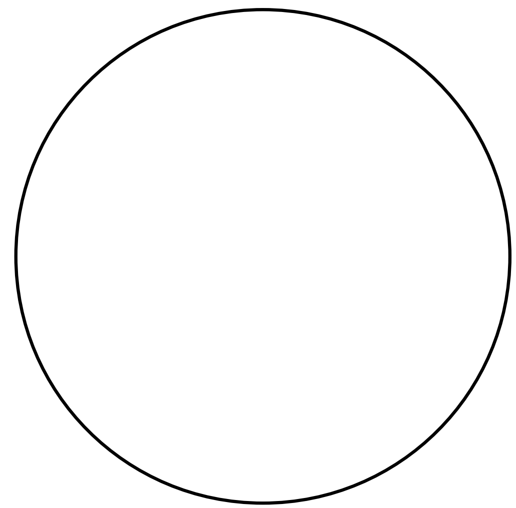
$X^{0.3}$





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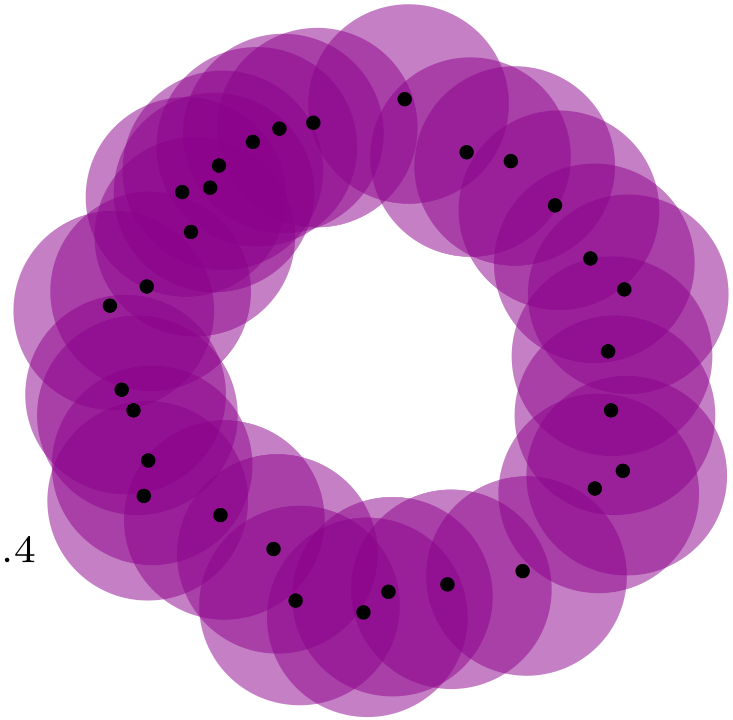
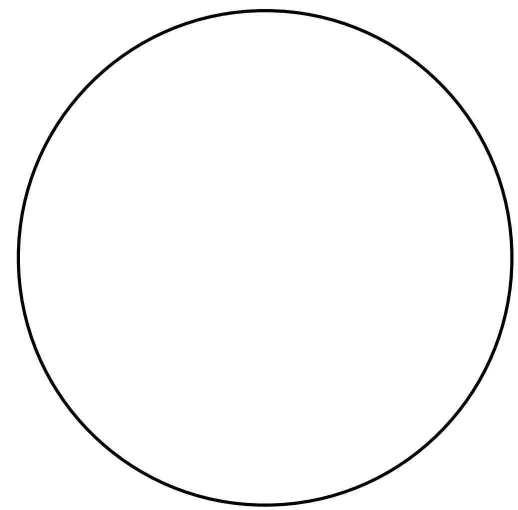

$X^{0.4}$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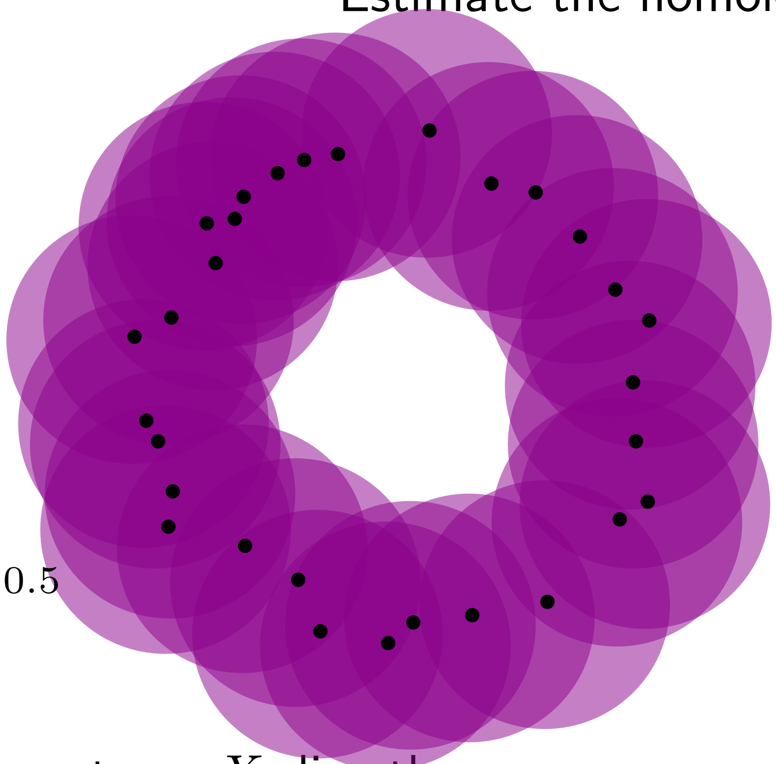
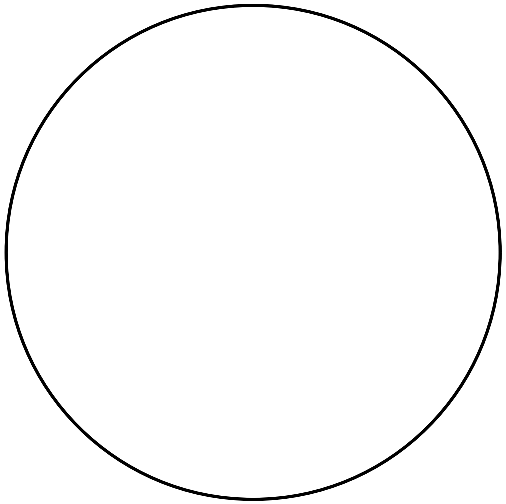


$X^{0.5}$



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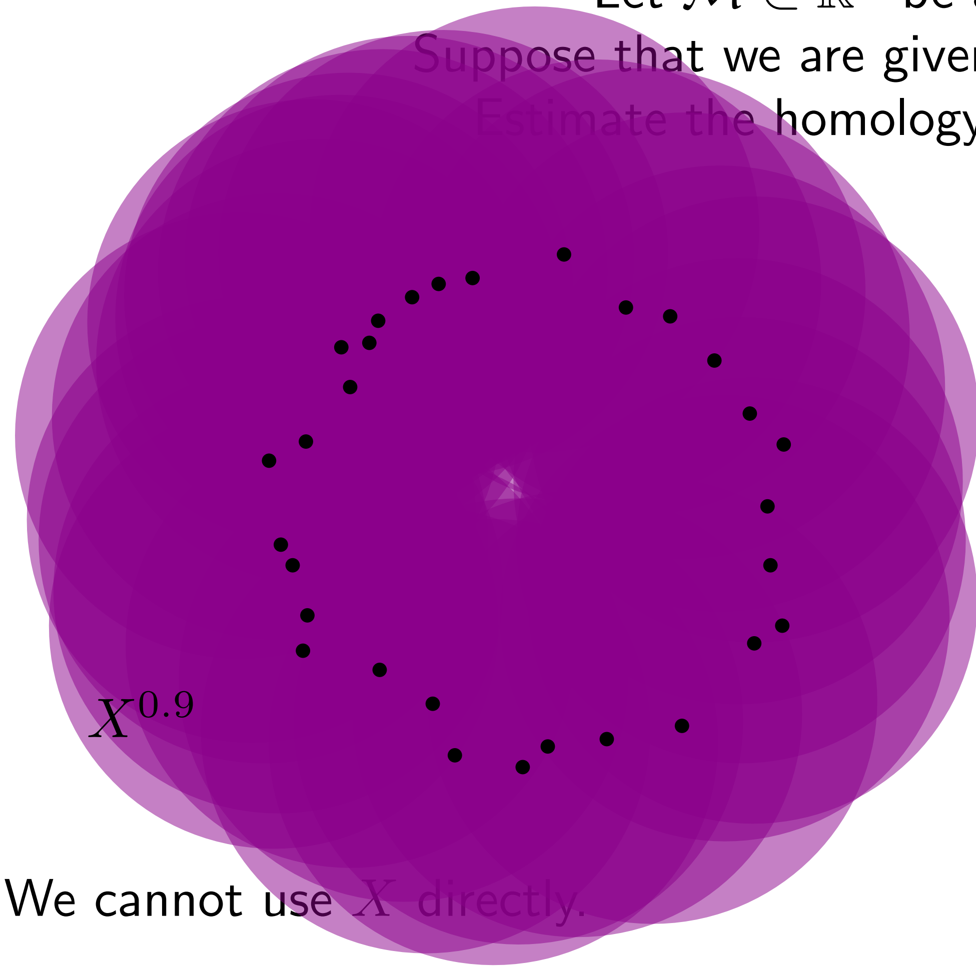

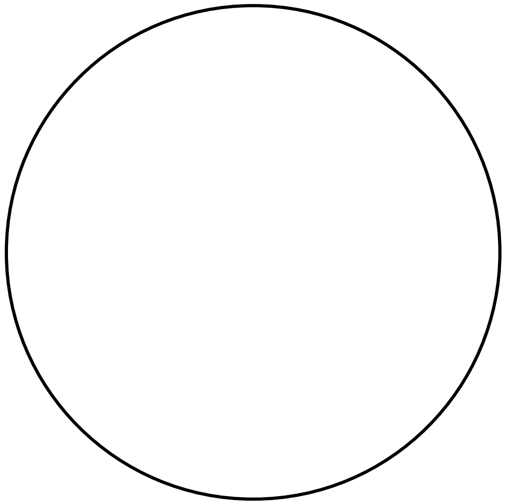

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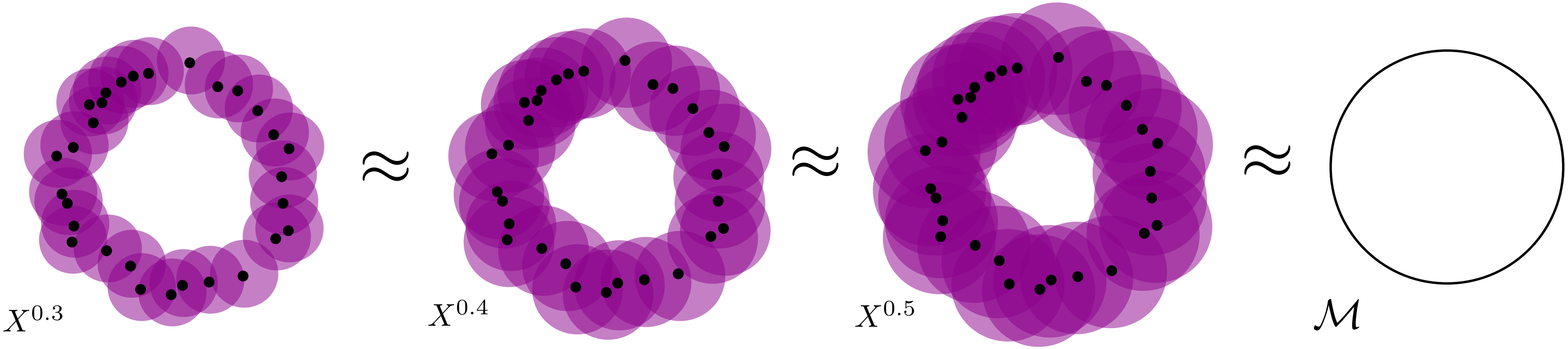
Some thickenings are homotopy equivalent to $\mathcal{M}$.



Hence we can recover the homology of $\mathcal{M}$:

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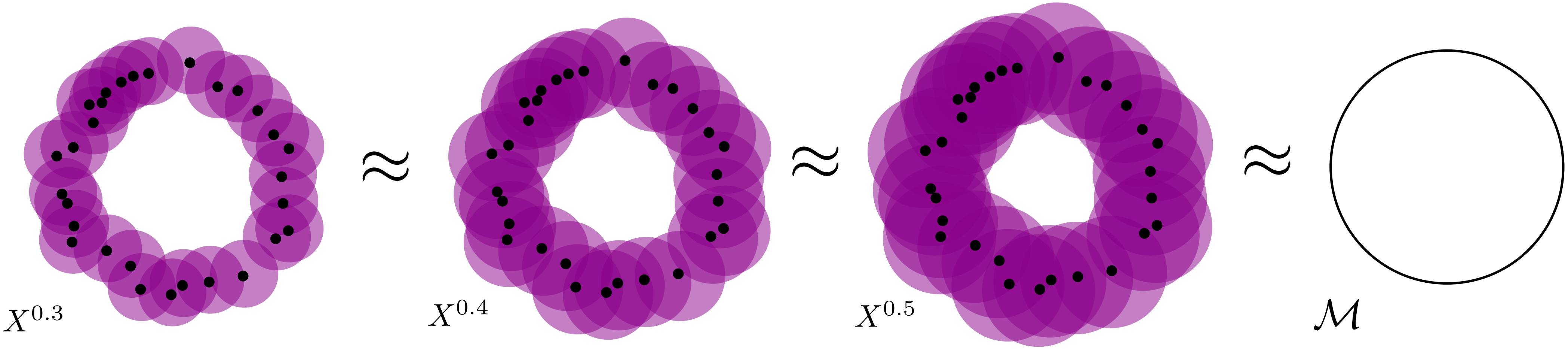

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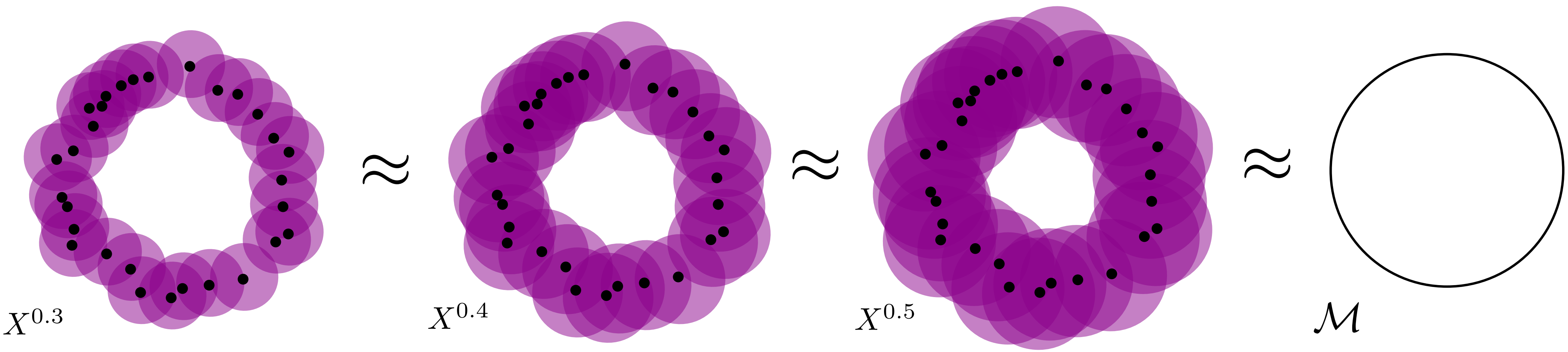

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Question 1: How to select a t such that $X^t \approx \mathcal{M}$? → Hausdorff distance → Reach

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Distância de Hausdorff

Let X be any subset of $\mathbb{R}^n$. The function **distance to** X is the map

$$\begin{aligned} \operatorname{dist}(\cdot, X) : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ y &\longmapsto \operatorname{dist}(y, X) = \inf\{\|y - x\|, x \in X\} \end{aligned}$$

A **projection** of $y \in \mathbb{R}^n$ on X is a point $x \in X$ which attains this infimum.

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$$\begin{aligned} \mathrm{d_H}(X, Y) &= \max\left\{\sup_{y \in Y} \operatorname{dist}(y, X), \ \sup_{x \in X} \operatorname{dist}(x, Y)\right\} \\ &= \max\left\{\sup_{y \in Y} \inf_{x \in X} \|x - y\|, \ \sup_{x \in X} \inf_{y \in Y} \|x - y\|\right\}. \end{aligned}$$

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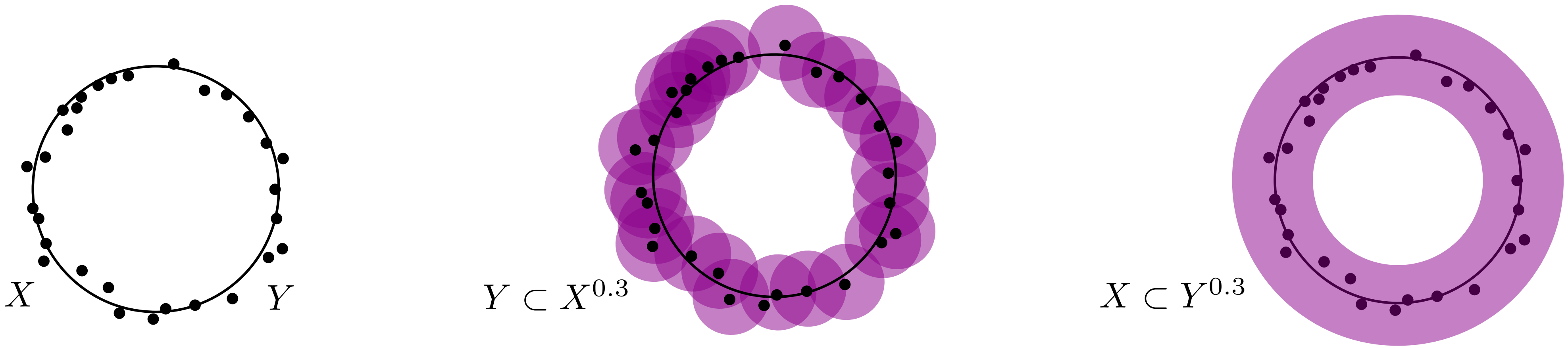
Proposition: The Hausdorff distance is equal to $\inf\{t \geq 0 \mid X \subset Y^t \text{ and } Y \subset X^t\}$.



Medial axis e reach

The **medial axis** of X is the subset $\operatorname{med}(X) \subset \mathbb{R}^n$ which consists of points $y \in \mathbb{R}^n$ that admit at least two projections on X:

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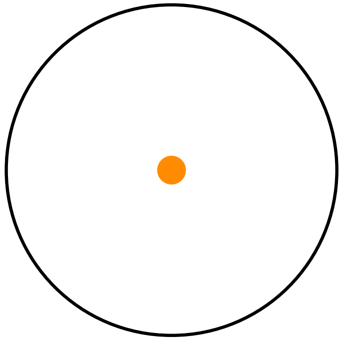
Examples:
The medial axis of the unit circle is the origin



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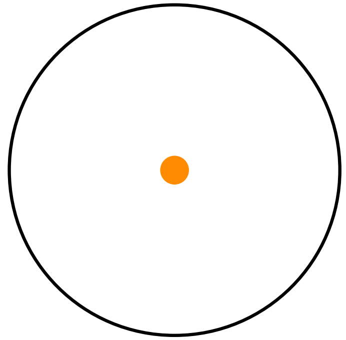
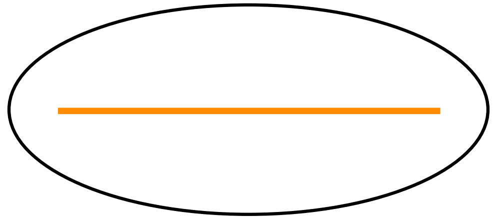


The medial axis of an ellipse is a segment



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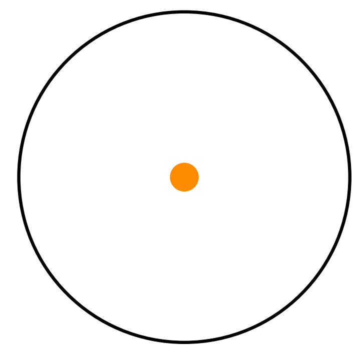
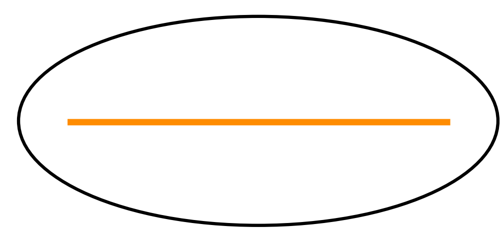
Examples:

The medial axis of the unit circle is the origin



The medial axis of an ellipse is a segment



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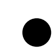

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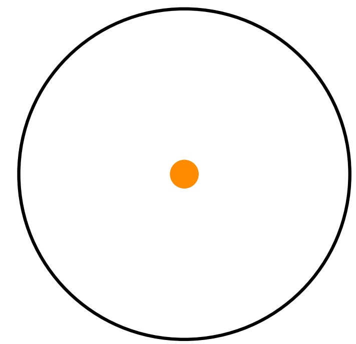

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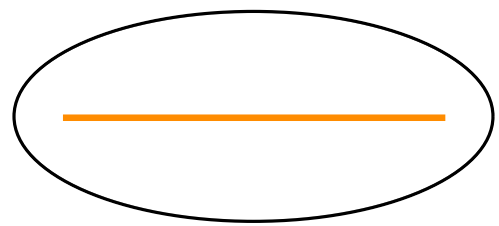

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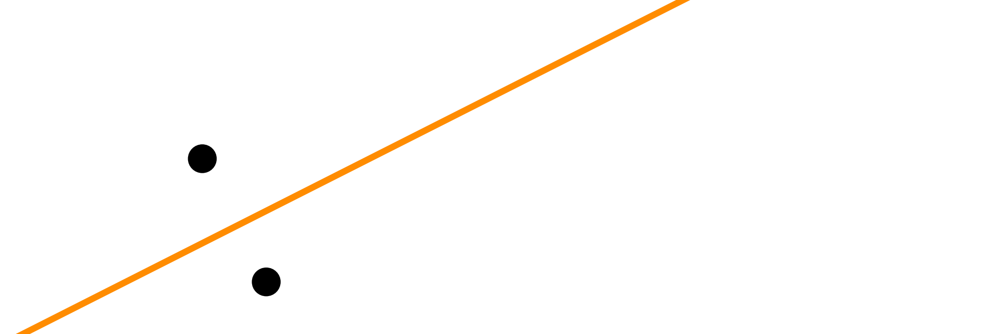

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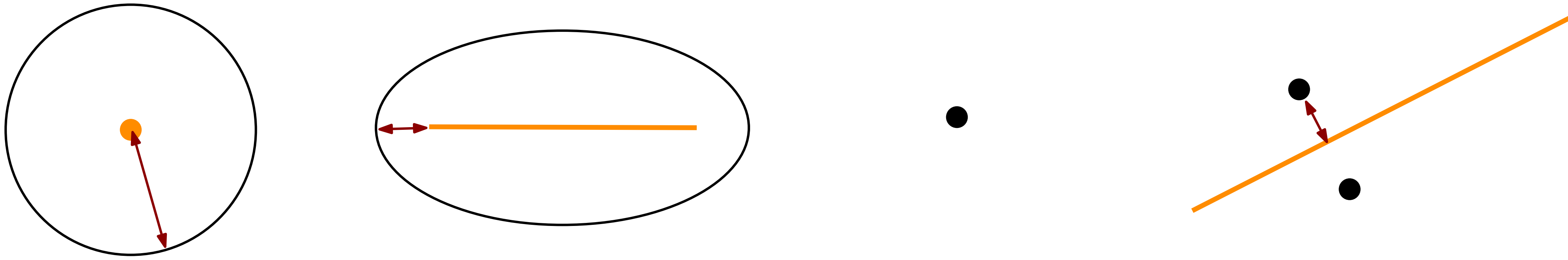

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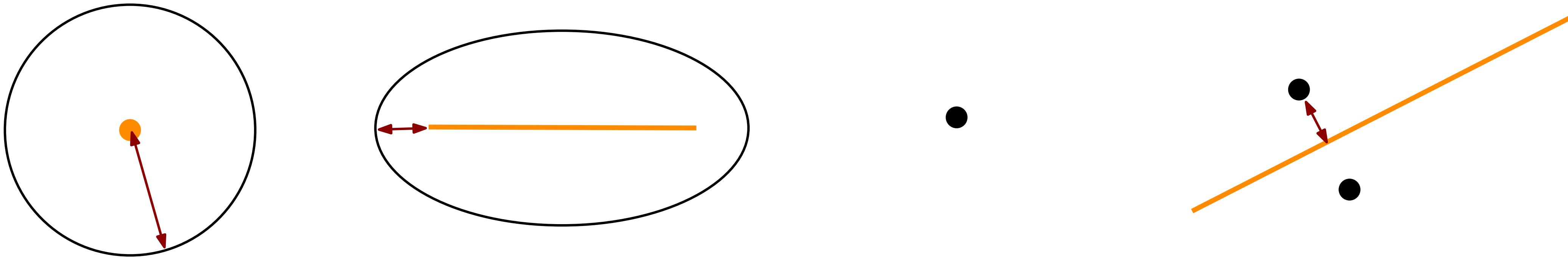

Proposition: For every $t \in [0, \mathrm{reach}\,(X))$, the spaces X and X^t are homotopy equivalent.

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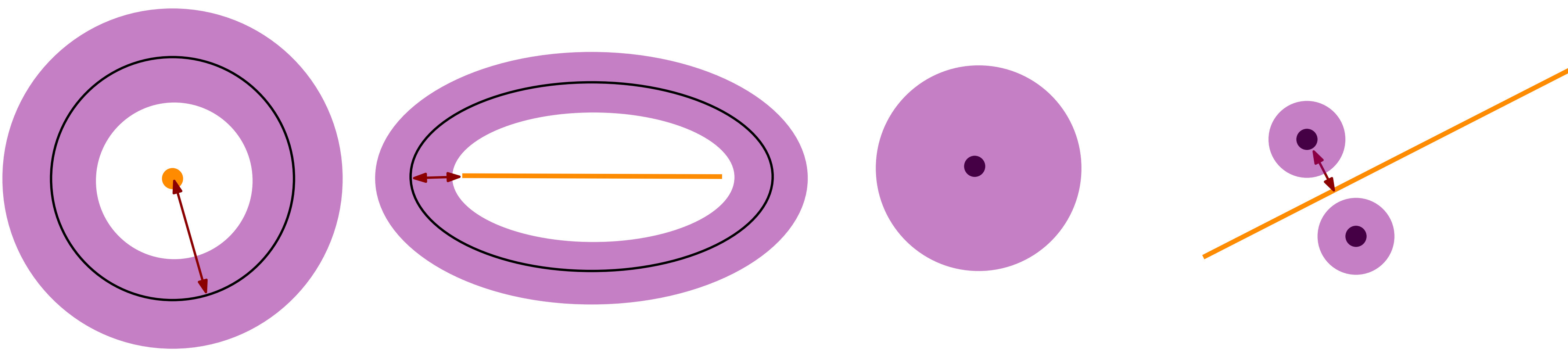

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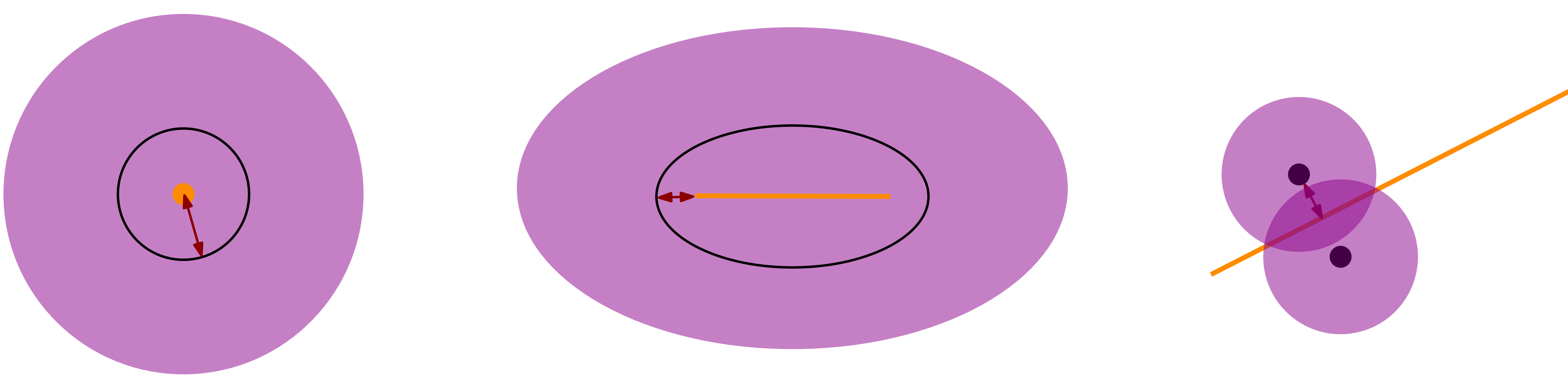

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If $t \geq \operatorname{reach}(X)$, the sets X and X^t may not be homotopy equivalent.

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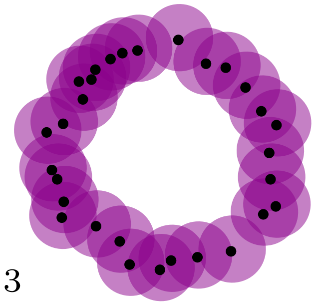
Proof: For every $t \in [0, \mathrm{reach}\,(X))$, the thickening X^t deform retracts onto X. A homotopy is given by the following map:

$$\begin{aligned} X^t \times [0,1] &\longrightarrow X^t \\ (x,t) &\longmapsto (1-t)x + t \cdot \mathrm{proj}\,(x, X)\,. \end{aligned}$$

Indeed, the projection $\mathrm{proj}\,(x, X)$ is well defined (it is unique).

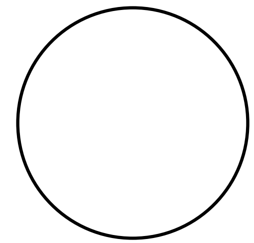
Seleção do parâmetro t

Remember Question 1: How to select a t such that $X^t \approx \mathcal{M}$?



$X^{0.3}$

$\approx$



$\mathcal{M}$

Theorem (Frédéric Chazal, David Cohen-Steiner, and André Lieutier, 2009):
Let X and $\mathcal{M}$ be subsets of $\mathbb{R}^n$. Suppose that $\mathcal{M}$ has positive reach, and that $\mathrm{d_H}\left(X, \mathcal{M}\right) \leq \frac{1}{17}\mathrm{reach}\left(\mathcal{M}\right)$.
Then X^t and $\mathcal{M}$ are homotopic equivalent, provided that

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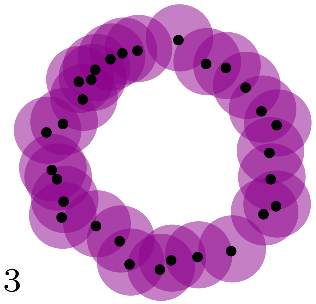

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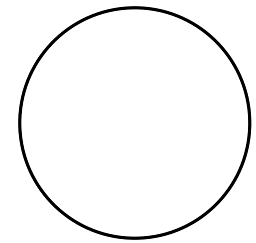

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Theorem (Partha Niyogi, Stephen Smale, and Shmuel Weinberger, 2008):
Let X and $\mathcal{M}$ be subsets of $\mathbb{R}^n$, with $\mathcal{M}$ a submanifold, and X a finite subset of $\mathcal{M}$.
Suppose that $\mathcal{M}$ has positive reach.
Then X^t and $\mathcal{M}$ are homotopic equivalent, provided that

$$t \in \left[2\mathrm{d_H}(X, \mathcal{M}), \sqrt{\frac{3}{5}}\mathrm{reach}(\mathcal{M})\right).$$

Triangulações (fracas)

Let us consider Question 2: How to compute the homology groups of X^t?

We must a triangulation of X^t, that is: a simplicial complex K homeomorphic to X.

Actually, we will define something weaker: a simplicial complex K that is homotopy equivalent to X.

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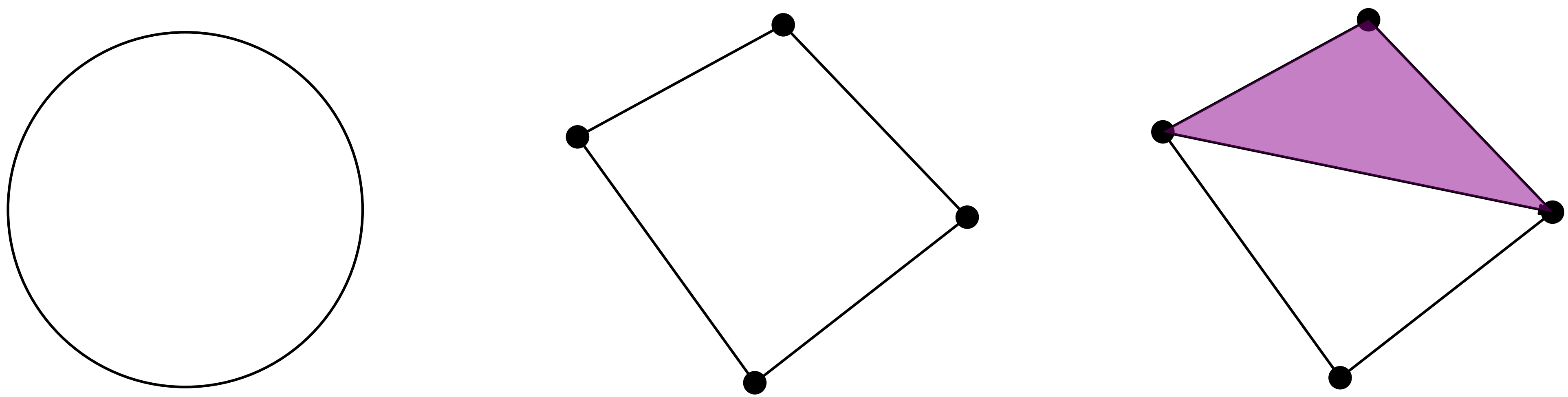
weak triangulation

Either case, we will have $\beta_i(X) = \beta_i(K)$ for all $i \geq 0$.



Definition: Let X be a topological space, and $\mathcal{U} = \{U_i\}_{1\leq i\leq N}$ a cover of X, that is, a collection of subsets $U_i \subset X$ such that

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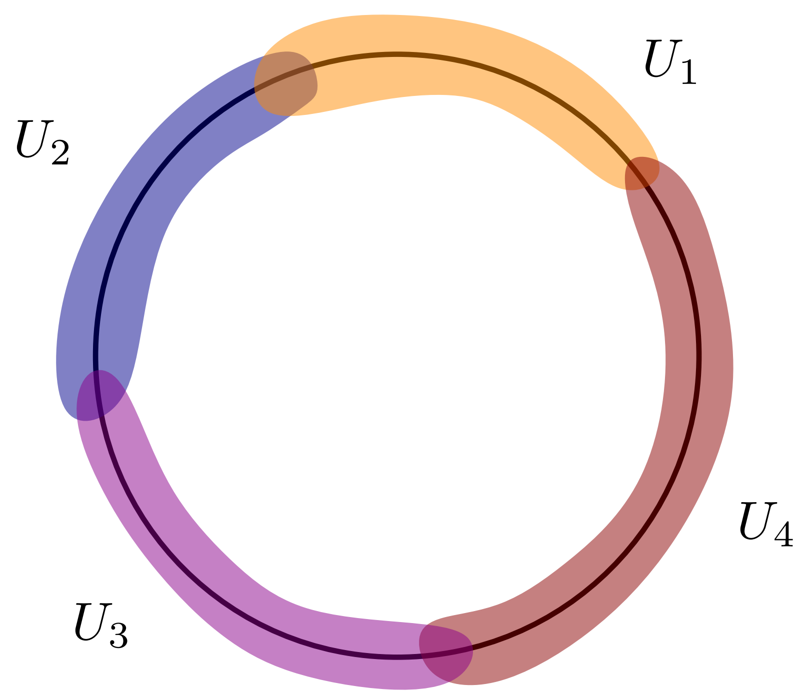
The **nerve** of $\mathcal{U}$ is the simplicial complex with vertex set $\{1, ..., N\}$ and whose m-simplices are the subsets $\{i_1, ..., i_m\} \subset \{1, ..., N\}$ such that $\bigcap_{k=0}^{m} U_{i_k} \neq \emptyset$. It is denoted $\mathcal{N}(\mathcal{U})$.



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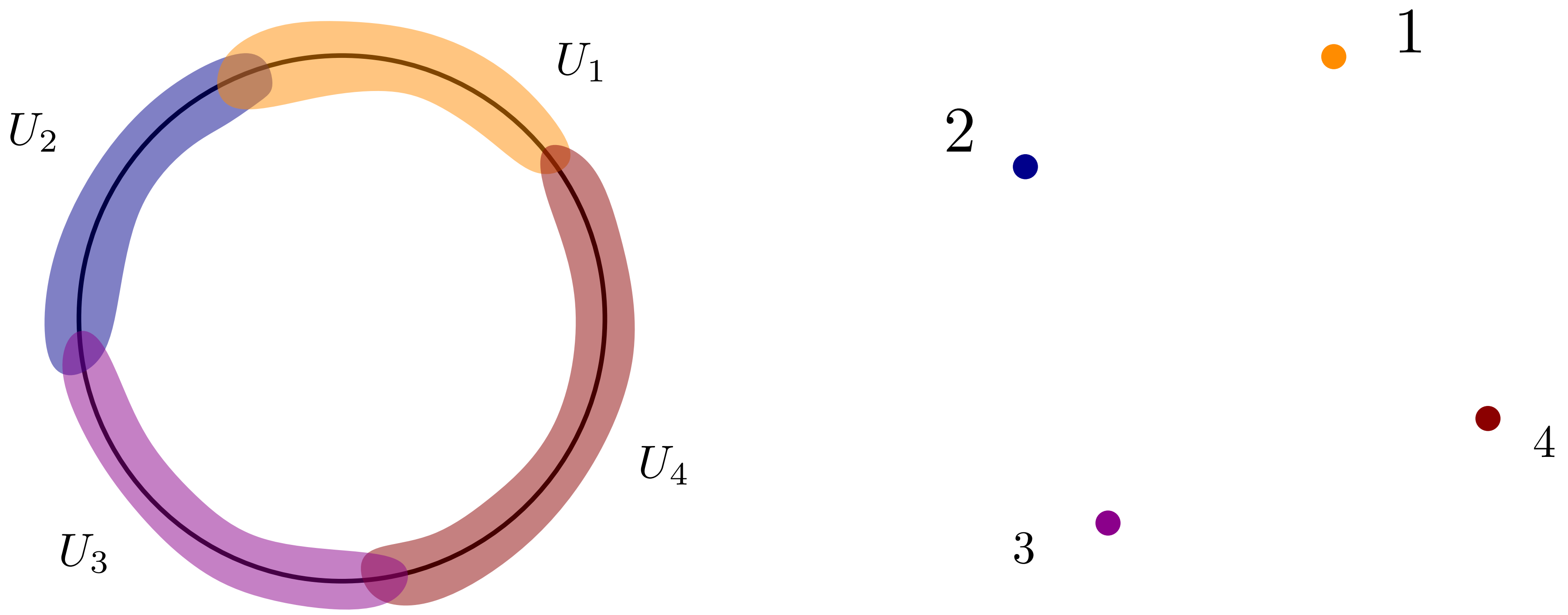

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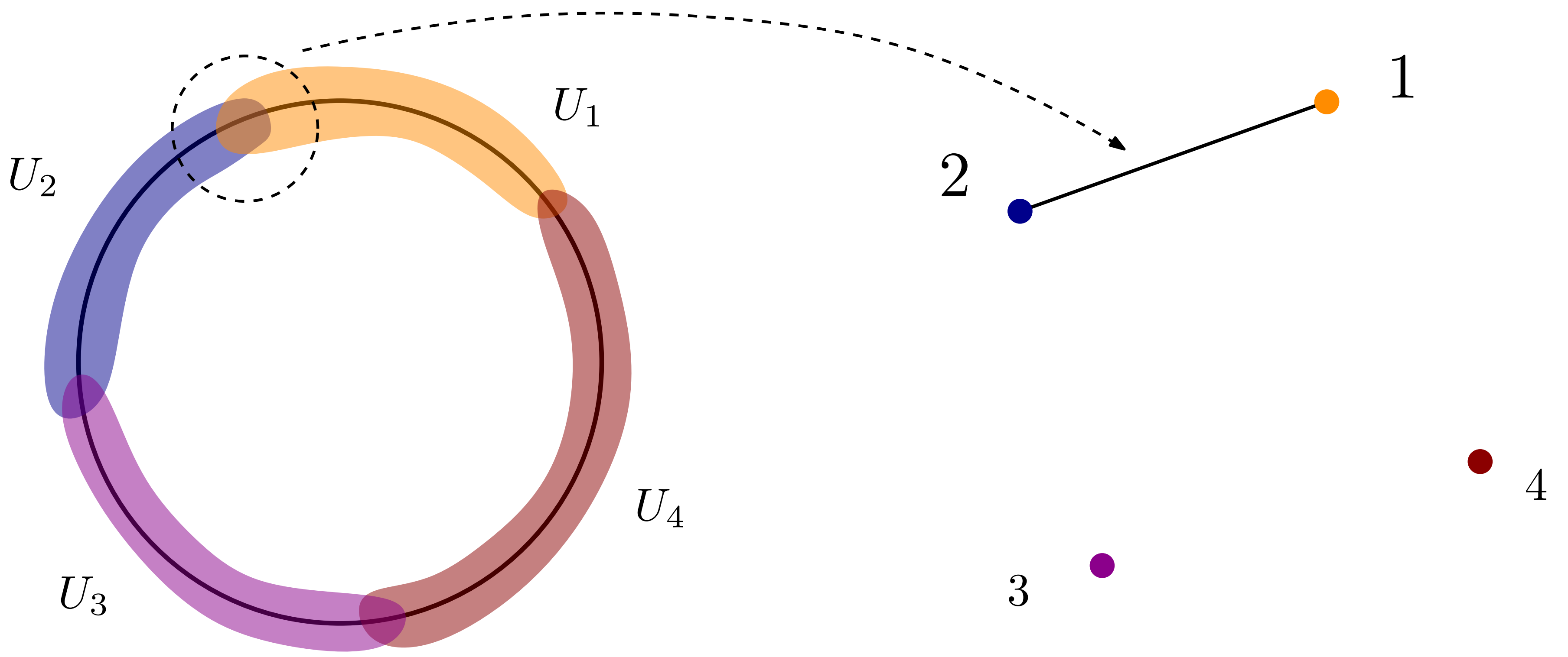

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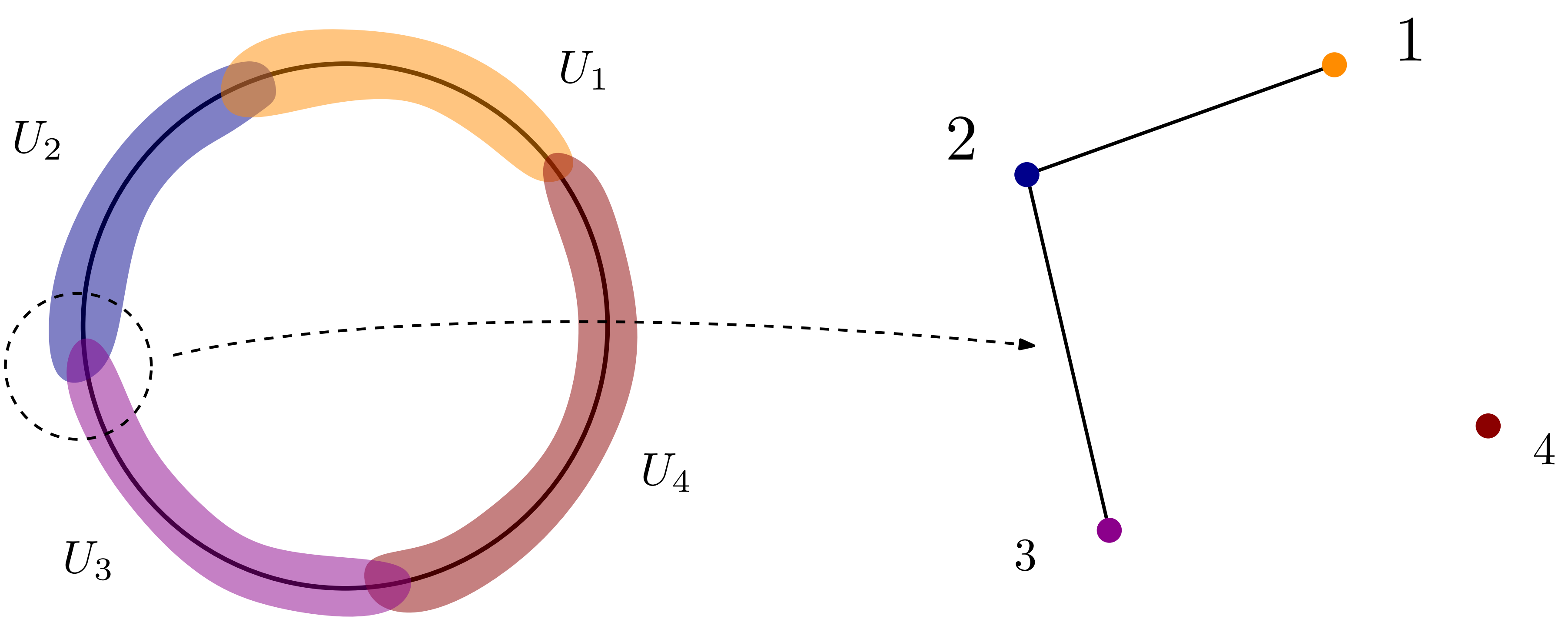

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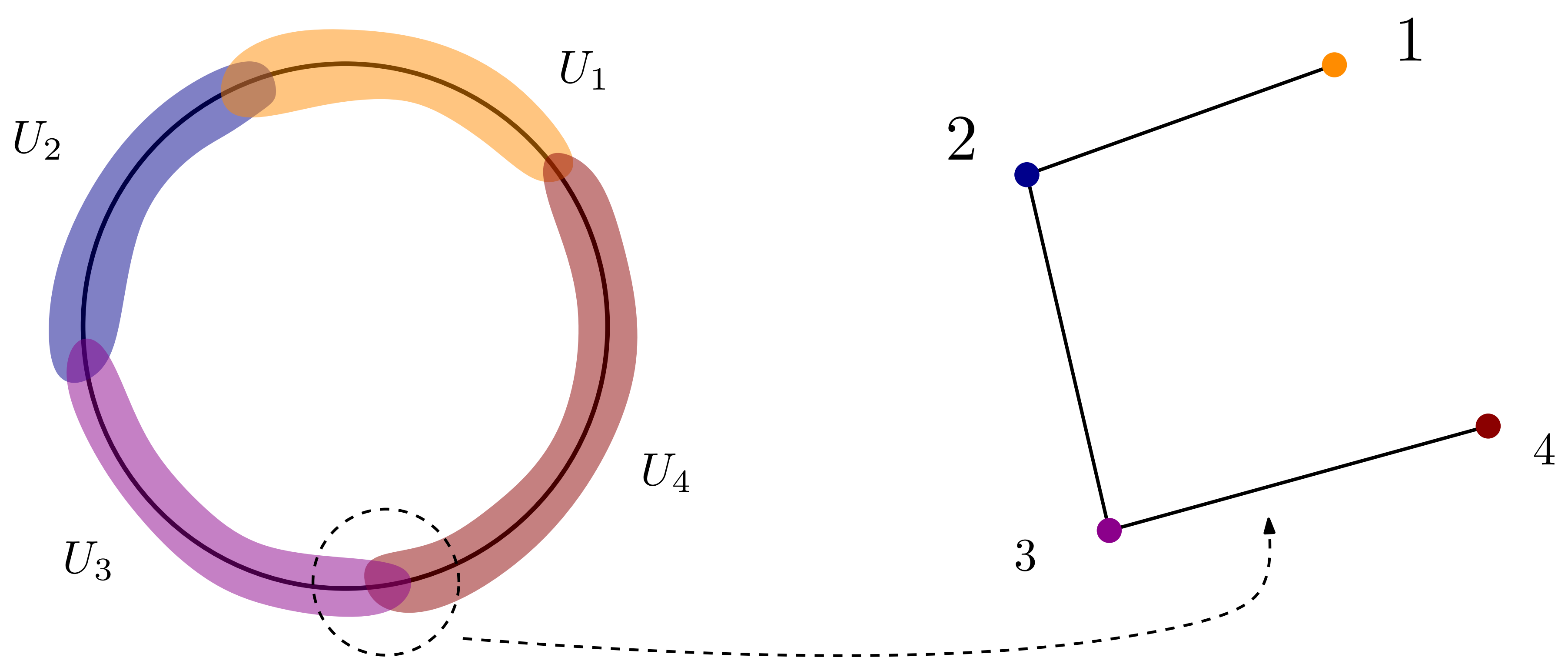

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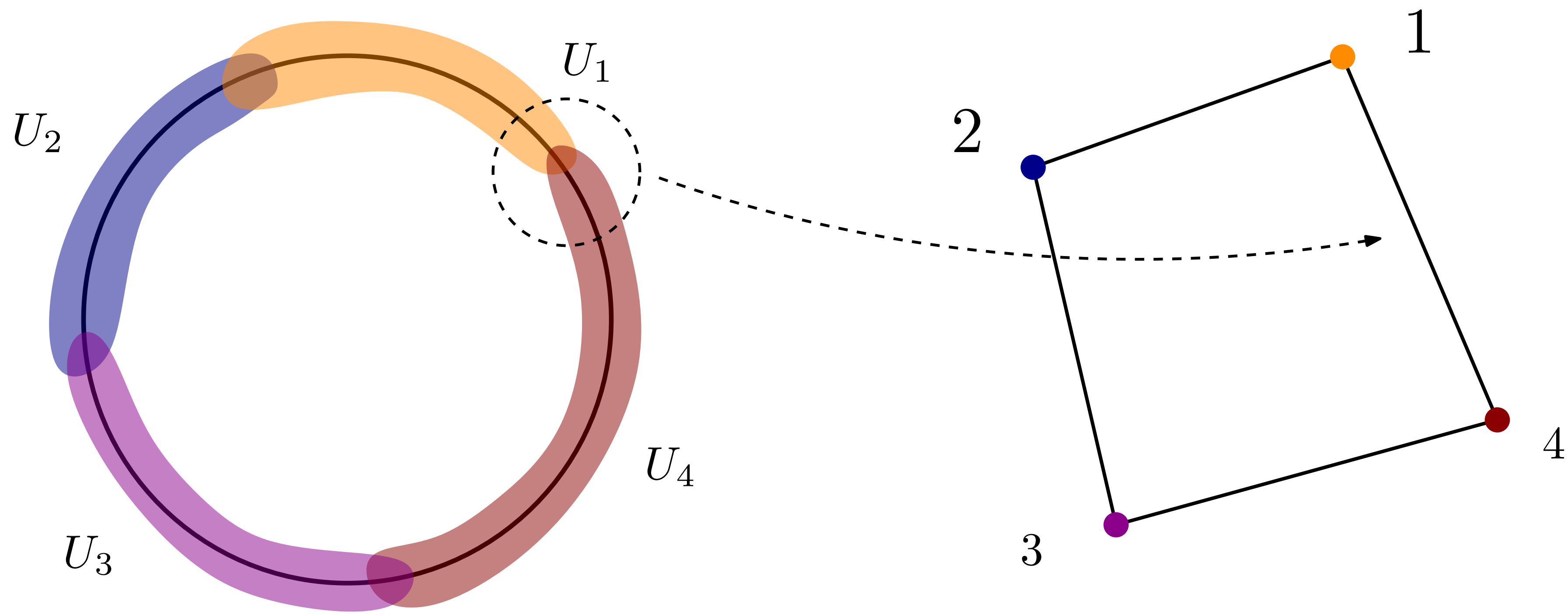

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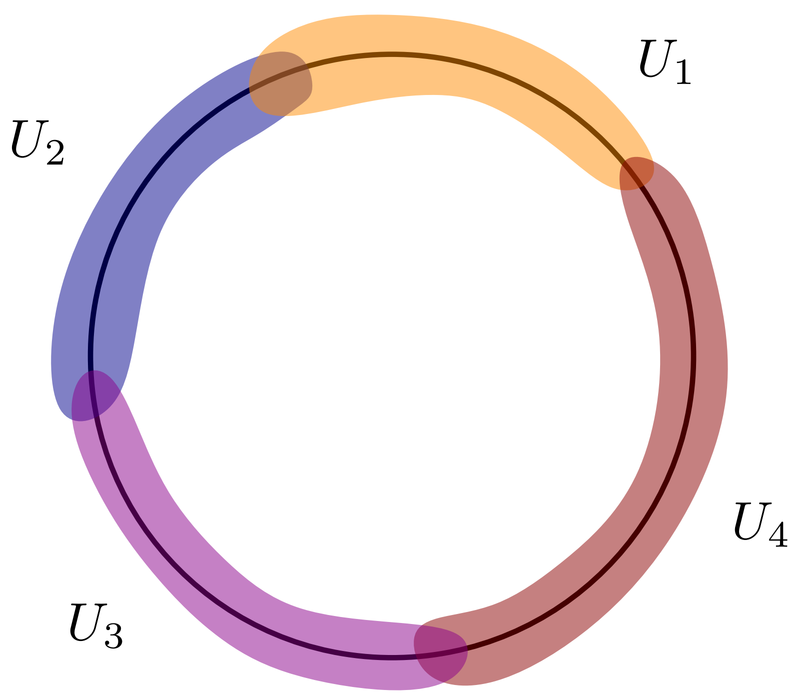

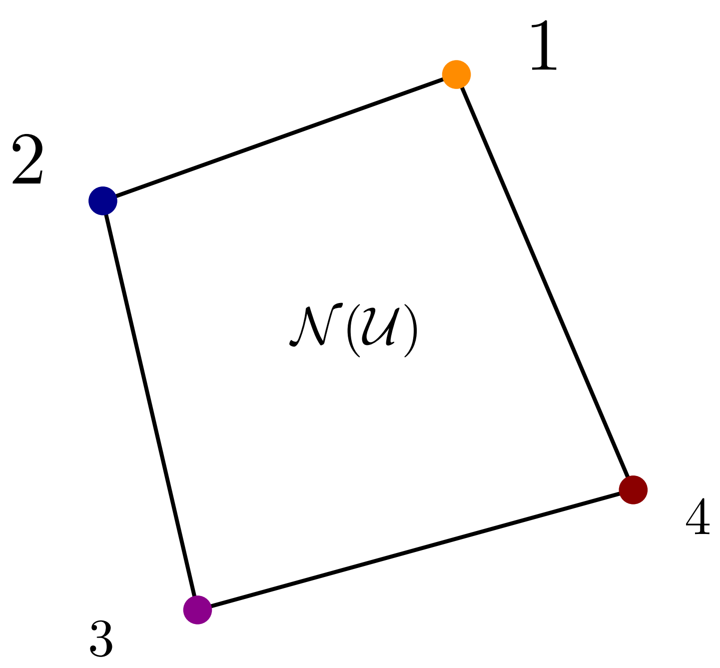

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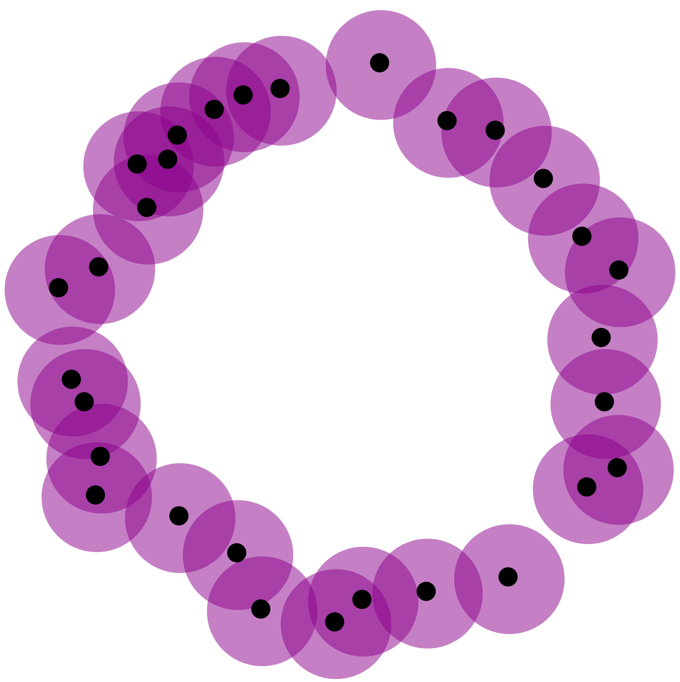

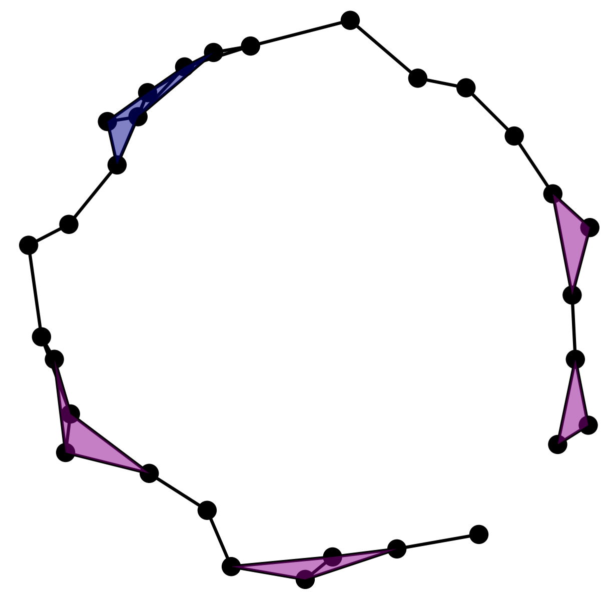

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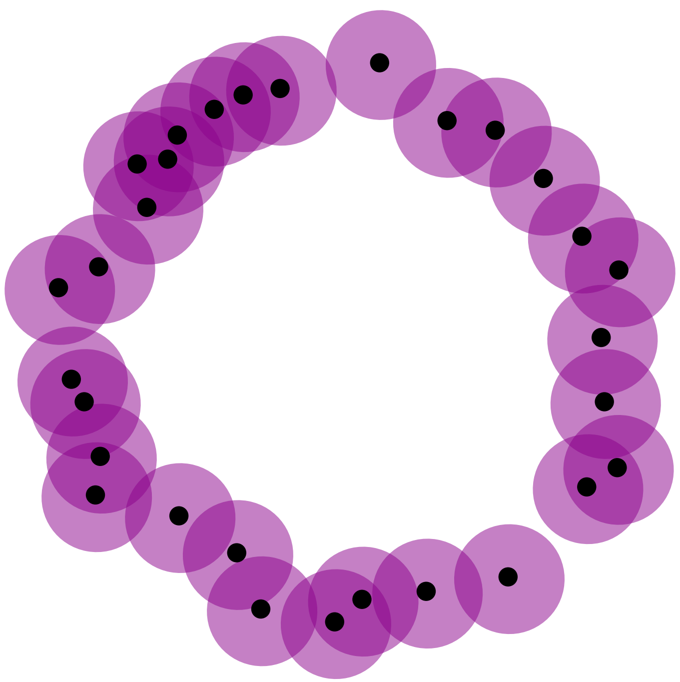

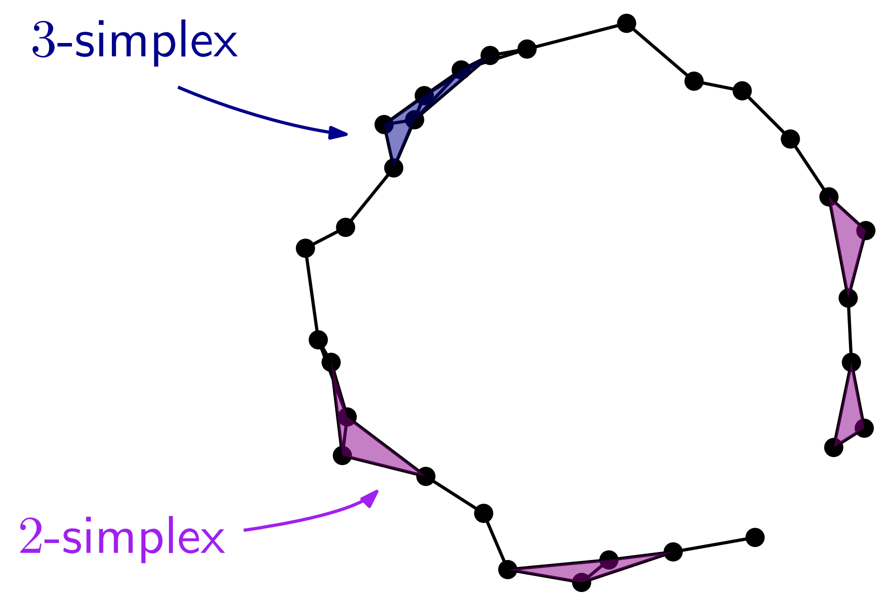

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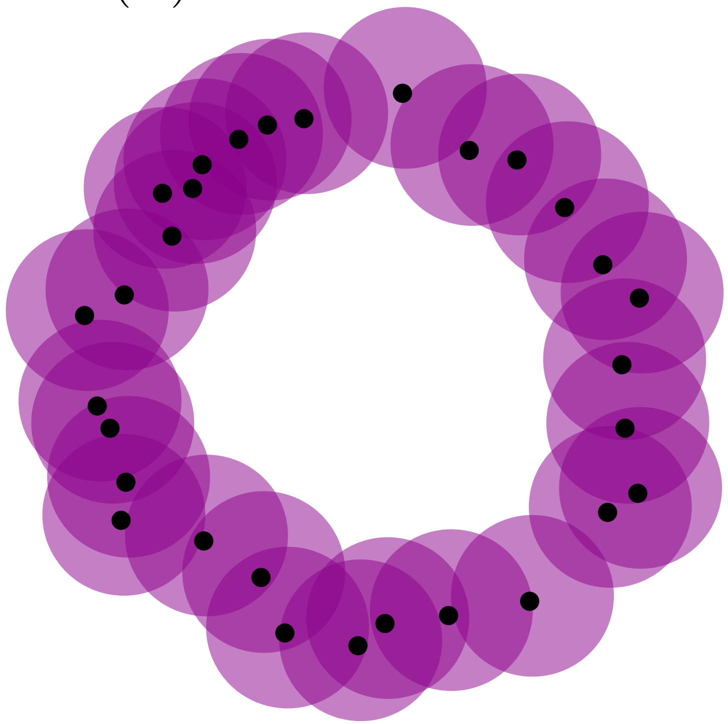

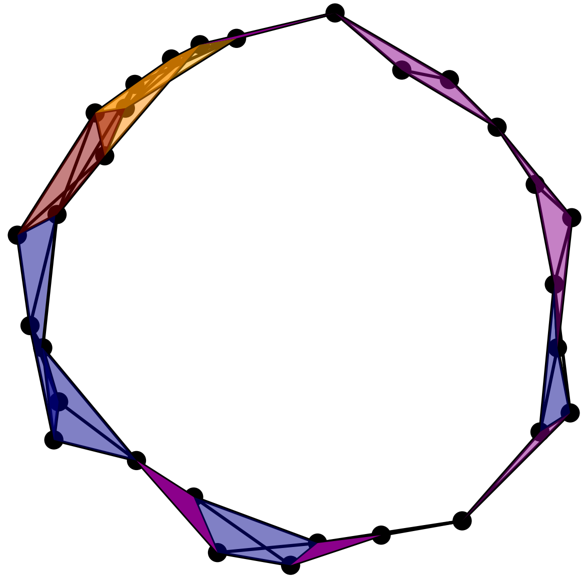

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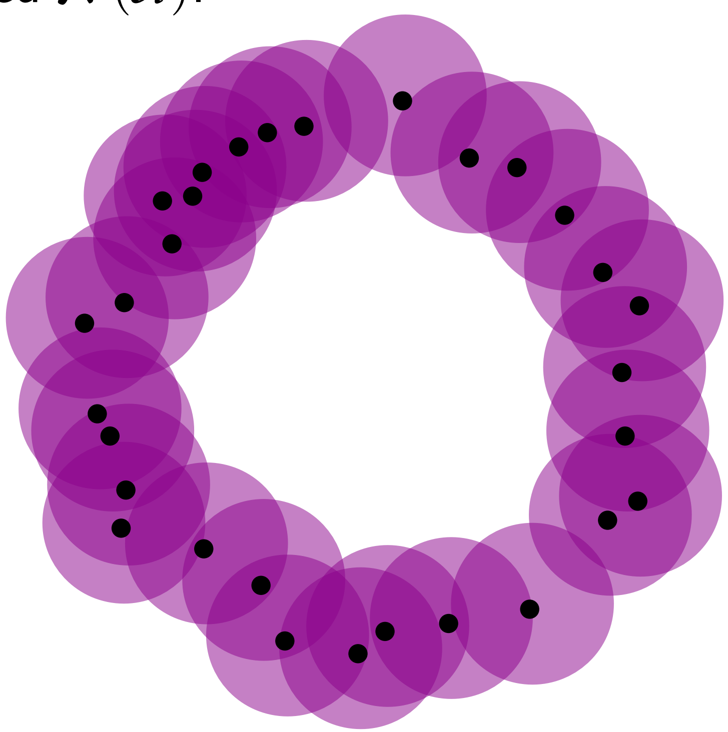

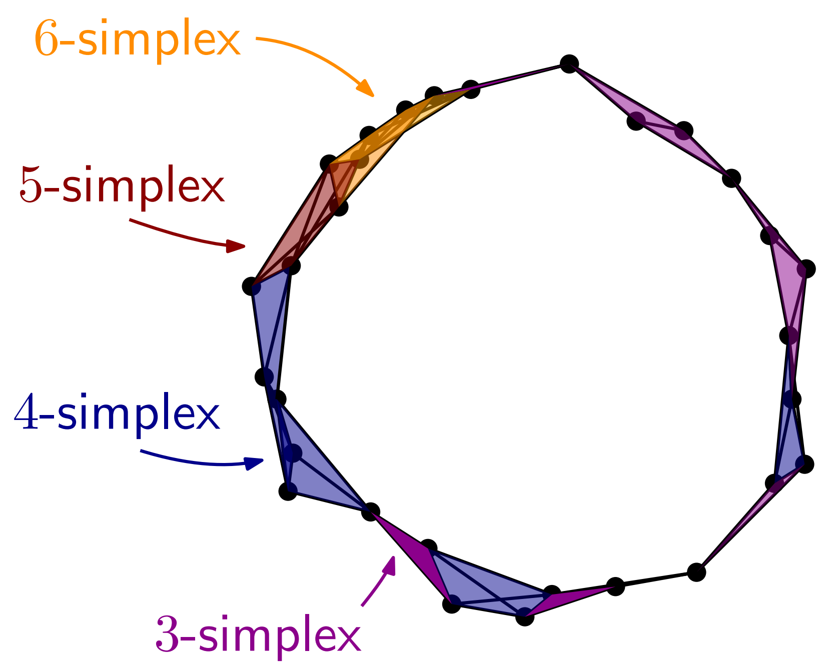

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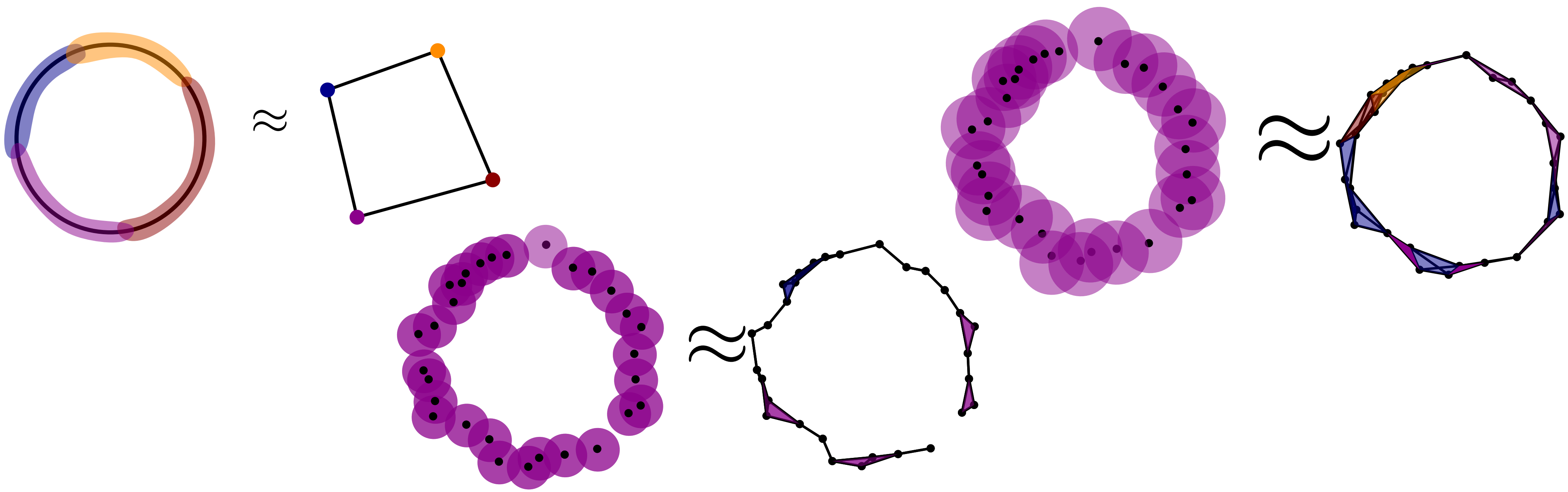
Nerve theorem: Consider $X \subset \mathbb{R}^n$. Suppose that each U_i are balls (or more generally, closed and convex). Then $\mathcal{N}(\mathcal{U})$ is homotopy equivalent to X.

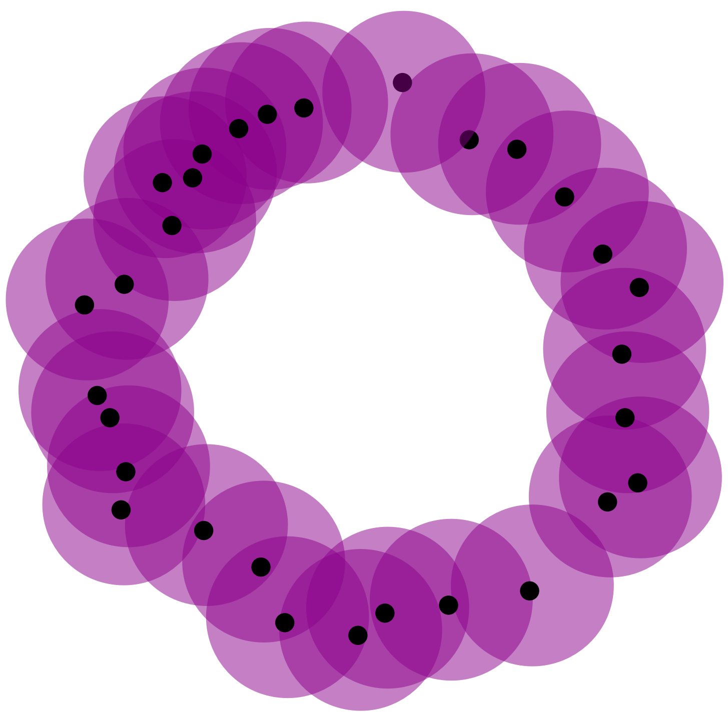


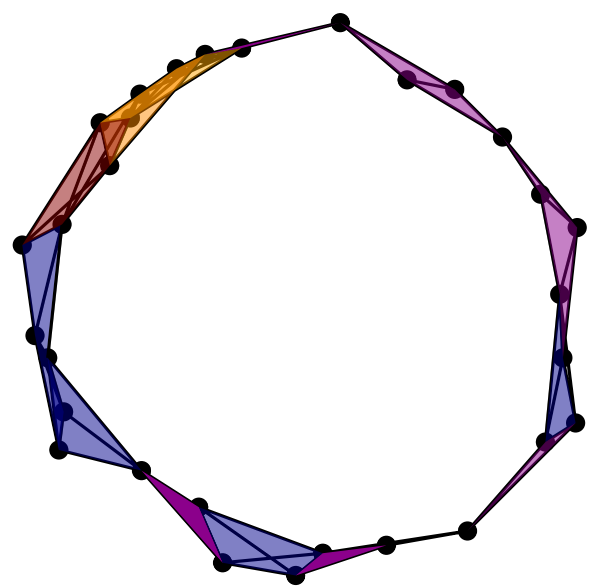
Let X be a finite subset of $\mathbb{R}^n$, and $t \geq 0$. Consider the collection

$$\mathcal{V}^t = \left\{\overline{\mathcal{B}}\left(x,t\right), x \in X\right\}.$$

This is a cover of the thickening X^t, and each components are closed balls. By Nerve Theorem, its nerve $\mathcal{N}(\mathcal{V}^t)$ has the homotopy type of X^t.

Definition: This nerve is denoted $\check{\mathrm{C}}\mathrm{ech}^t(X)$ and is called the **Čech complex** of X at time t.





Let X be a finite subset of $\mathbb{R}^n$, and $t \geq 0$. Consider the collection

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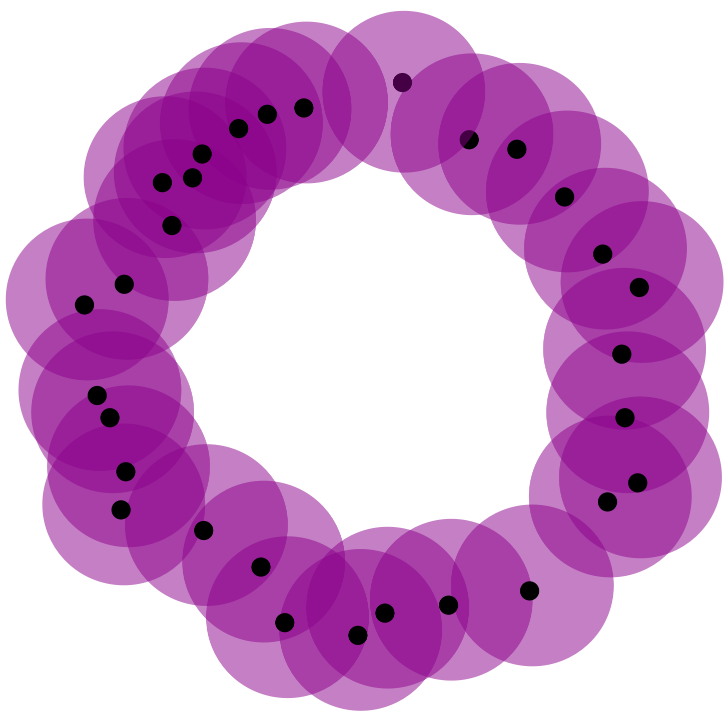

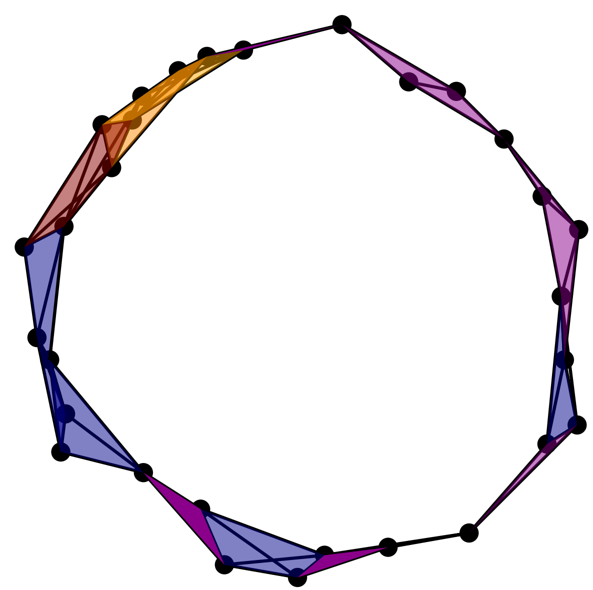

→ The Question 2 (How to compute the homology groups of X^t?) is solved.

Computação do complexo de Čech

Let $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$ be finite, let $t \geq 0$ and consider the t-thickening

$$X^t = \bigcup_{x \in X} \overline{\mathcal{B}}\left(x, t\right).$$

By definition, its nerve, $\check{\mathrm{C}}\mathrm{ech}^t(X)$, the Čech complex at time t, is a simplicial complex on the vertices $\{1, \ldots, N\}$ whose simplices are the subsets $\{i_1, ..., i_m\}$ such that

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Therefore, computing the Čech complex relies on the following geometric predicate:

Given m closed balls of $\mathbb{R}^n$, do they intersect?

This problem is known as the *smallest circle problem*.
It can can be solved in $O(m)$ time, where m is the number of points.

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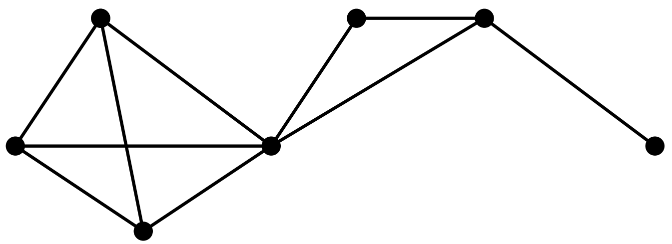
$\longrightarrow$ in practice, we prefer a more simple version

Complexo de clique

Let G be a graph.
We call a **clique** of G a set of vertices $v_1, ..., v_m$ such that for every $i, j \in [\![1, m]\!]$ with $i \neq j$, the edge $[v_i, v_j]$ belongs to G.



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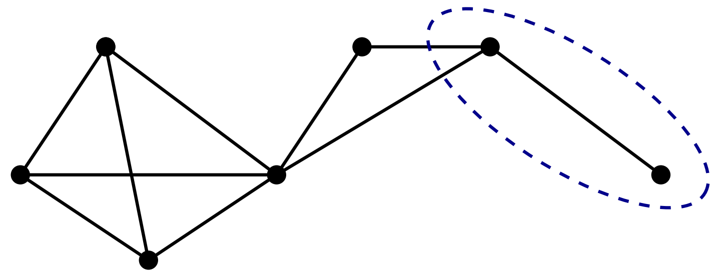

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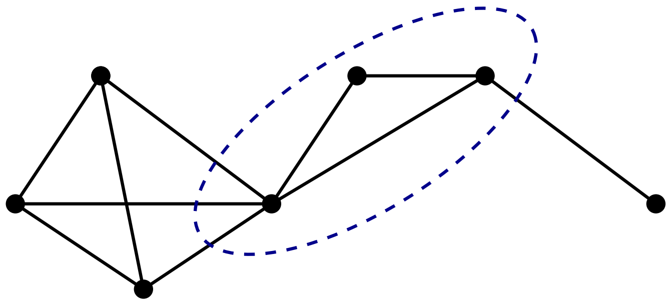

3-clique

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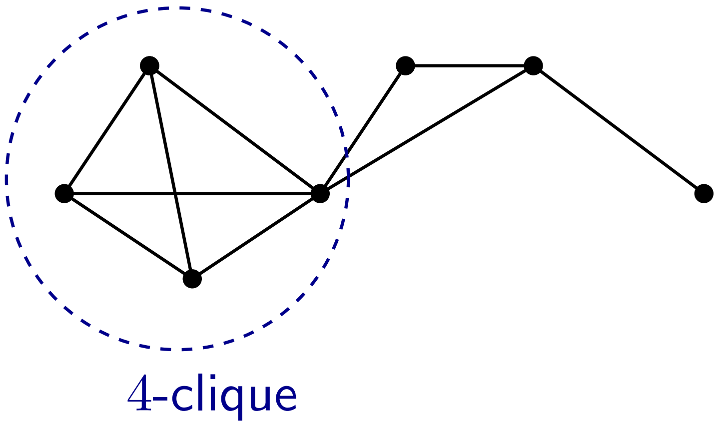

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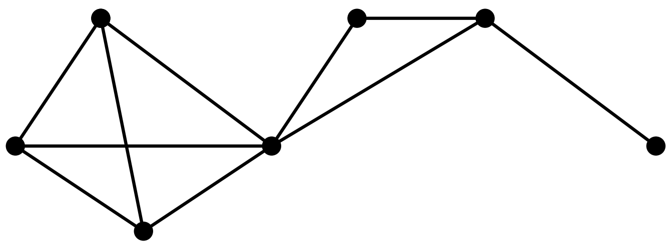
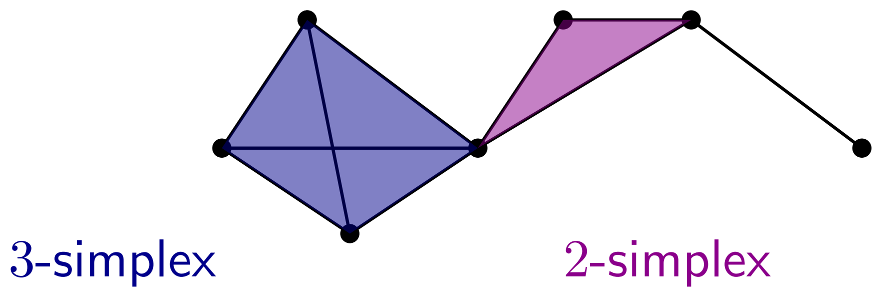


Definition: Given a graph G, the corresponding **clique complex** is the simplicial complex whose

- vertices are the vertices of G,
- simplices are the sets of vertices of the cliques of G.



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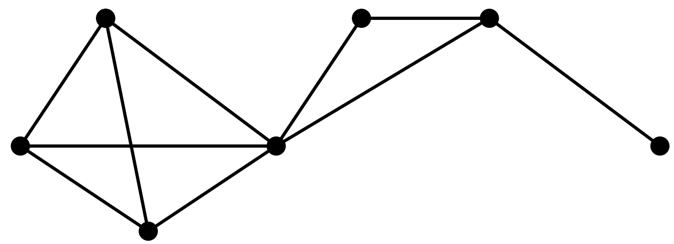

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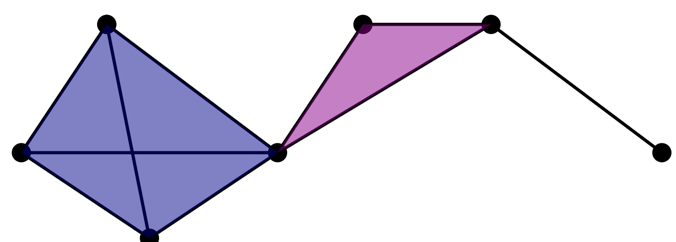


3-simplex 2-simplex

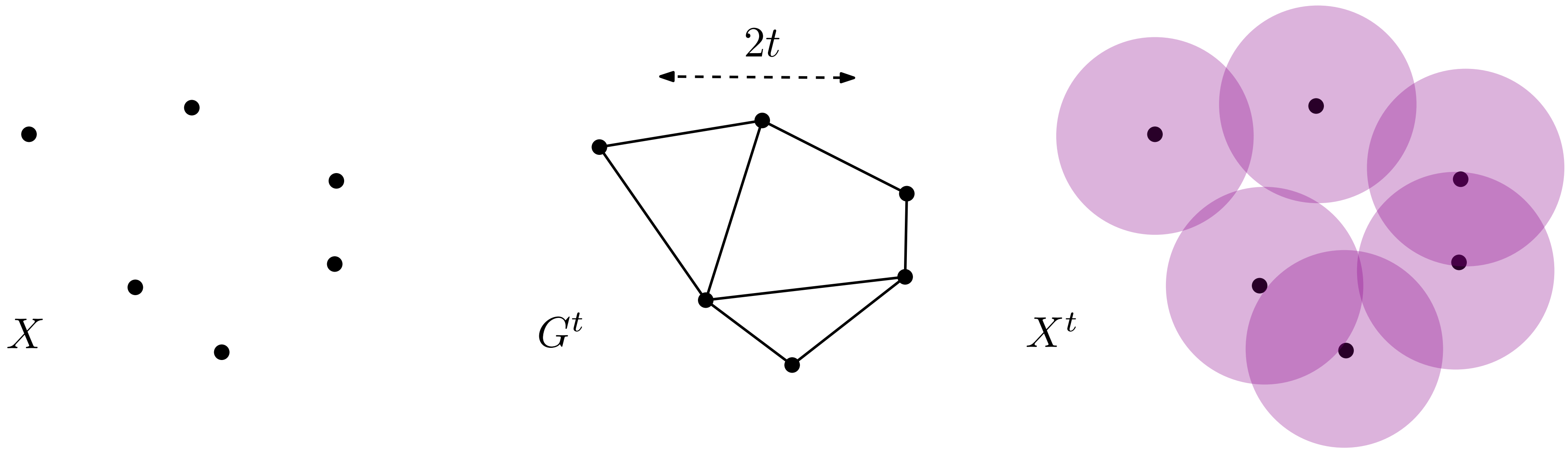
Observation: The clique complex of a graph is a simplicial complex.

Complexo de Rips

Let $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$ and $t \geq 0$.

Consider the graph G^t whose vertex set is $\{1, \ldots, N\}$, and whose edges are the pairs (i, j) such that $\|x_i - x_j\| \leq 2t$.

Alternatively, G^t can be seen as the 1-skeleton of the Čech complex $\check{\text{C}}\text{ech}^t(X)$.



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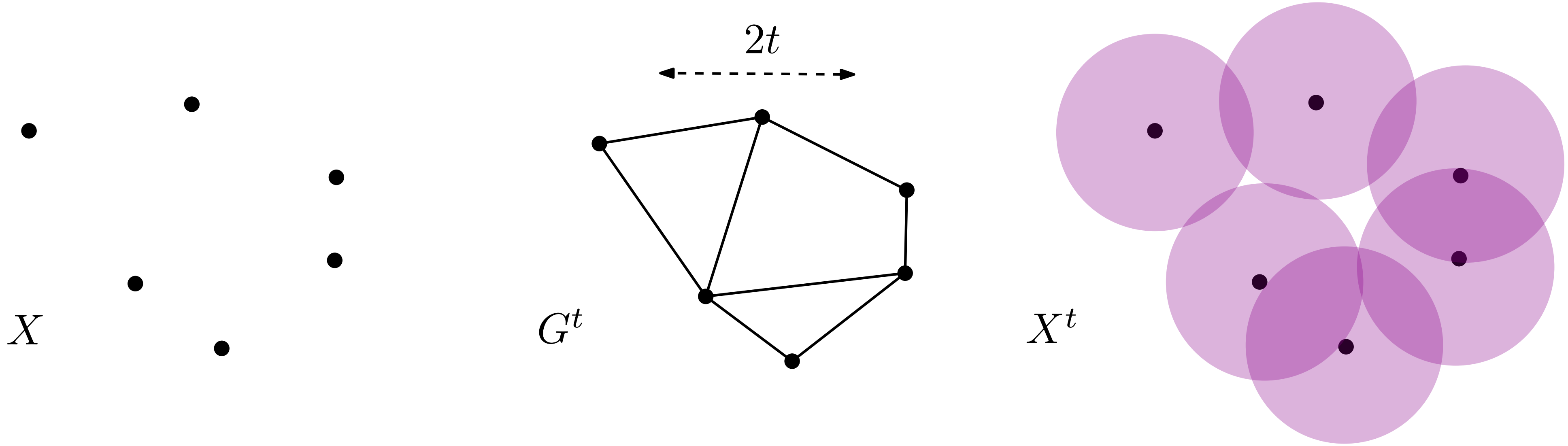
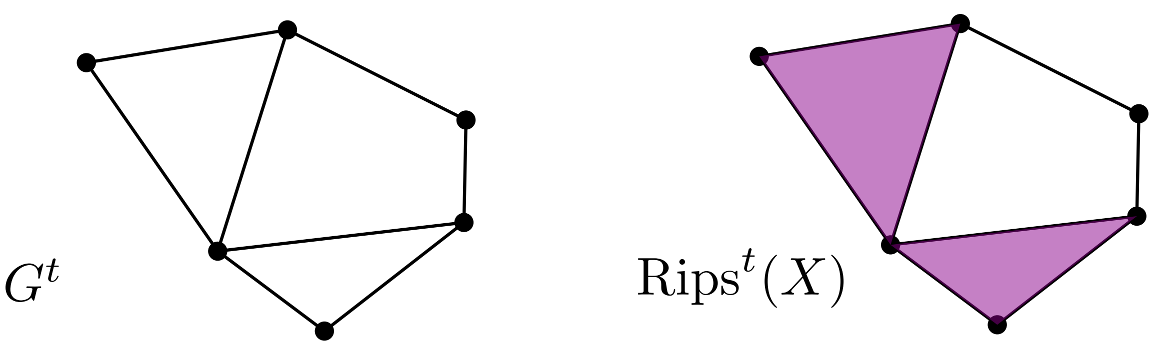


Definition: The **Rips complex** of X at time t is the clique complex of the graph G^t. We denote it $\mathrm{Rips}^t(X)$.



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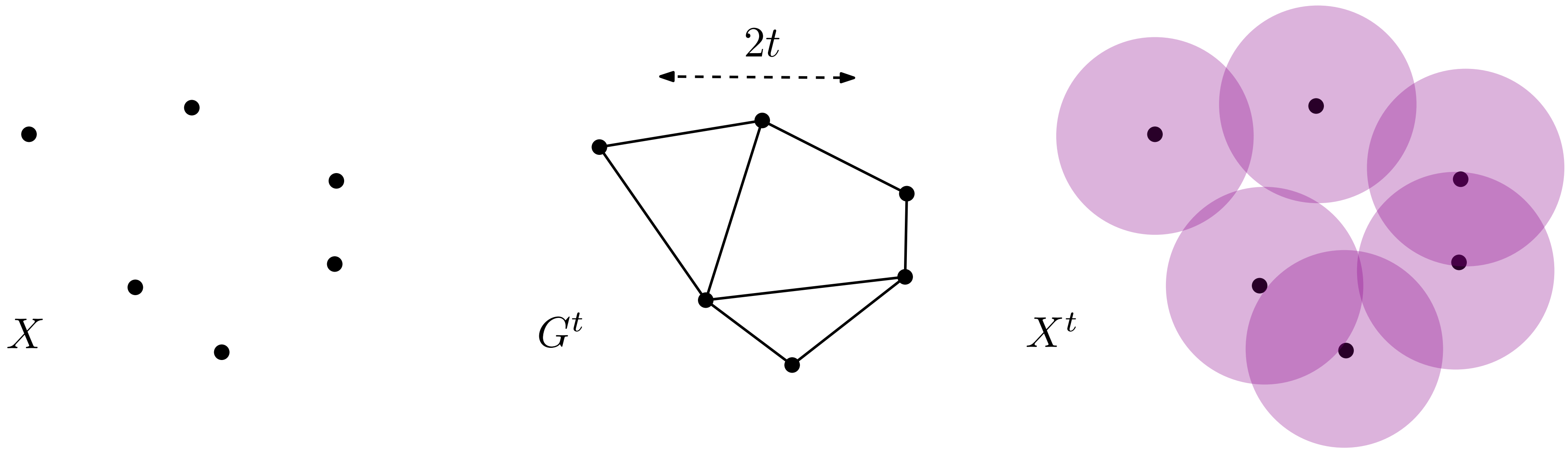

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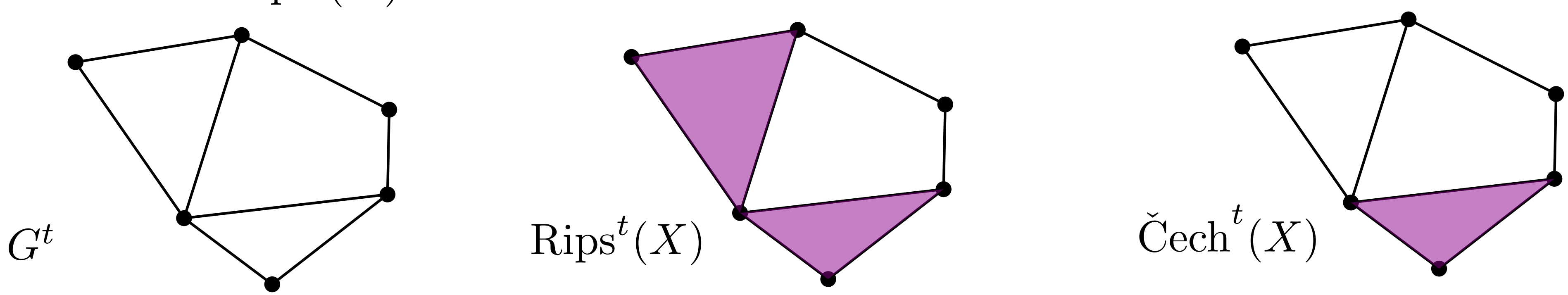

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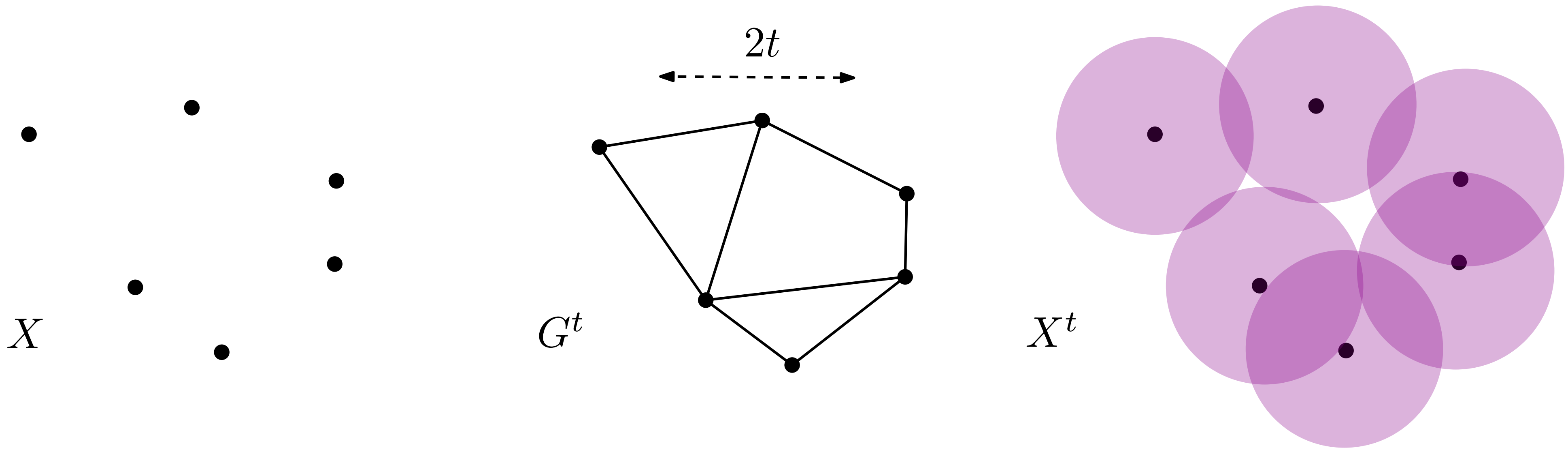

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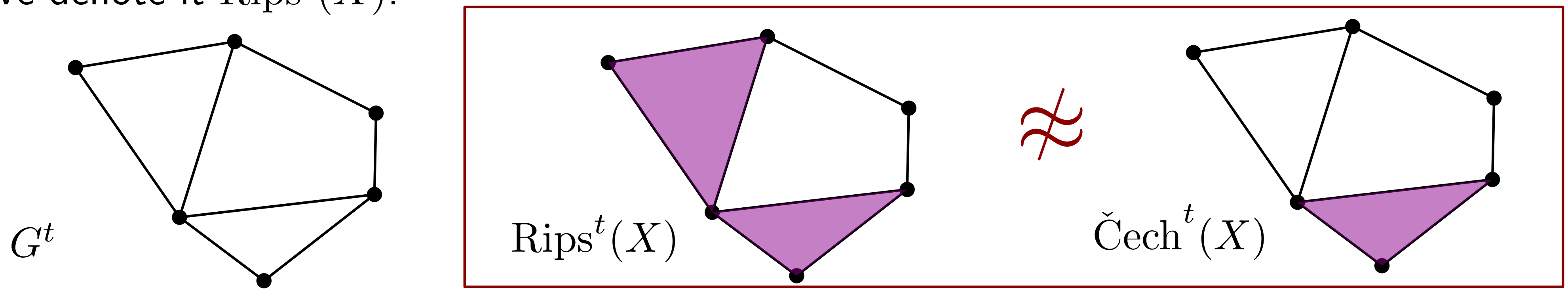
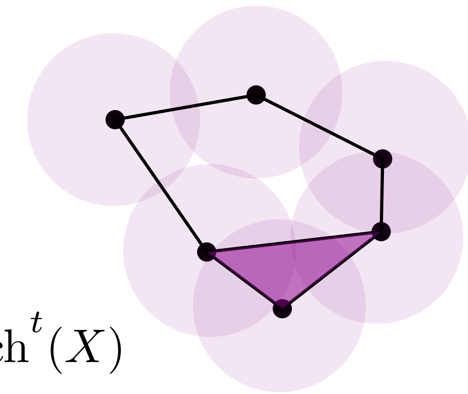


Complexo de Rips

Proposition: For every $t \geq 0$, we have

$$\check{\mathrm{C}}\mathrm{ech}^t(X) \subset \mathrm{Rips}^t(X) \subset \check{\mathrm{C}}\mathrm{ech}^{2t}(X).$$



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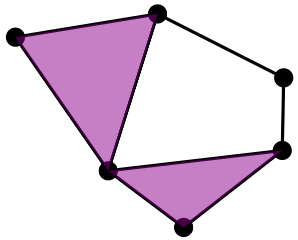

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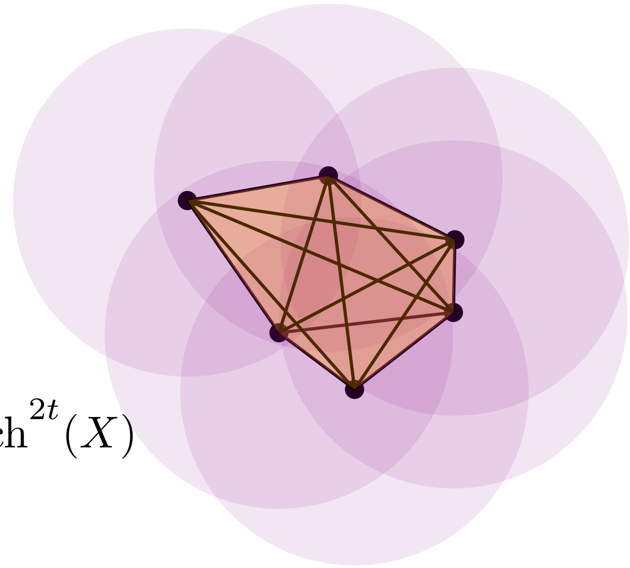

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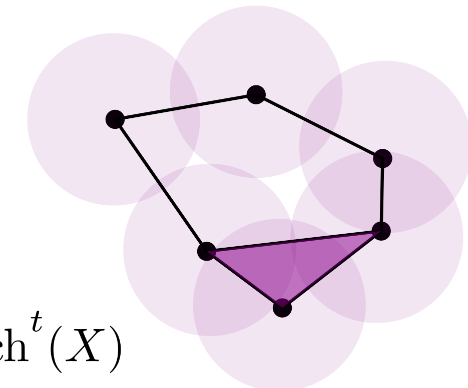

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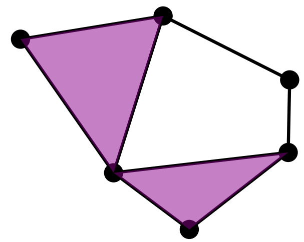

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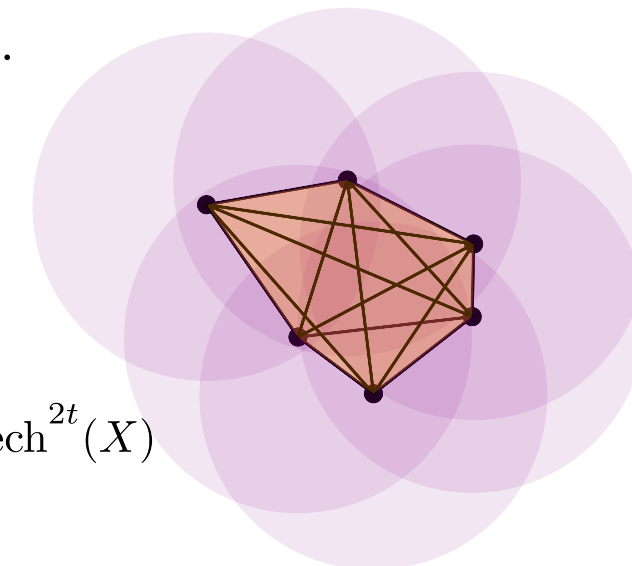

$\check{\mathrm{C}}\mathrm{ech}^{2t}(X)$

Proof: Let $t \geq 0$. The first inclusion follows from the fact that $\mathrm{Rips}^t(X)$ is the clique complex of $\check{\mathrm{C}}\mathrm{ech}^t(X)$.

To prove the second one, choose a simplex $\sigma \in \mathrm{Rips}^t(X)$. Let us prove that $\omega \in \check{\mathrm{C}}\mathrm{ech}^{2t}(X)$.

Let $x \in \sigma$ be any vertex. Note that $\forall y \in \sigma$, we have $\|x - y\| \leq 2t$ by definition of the Rips complex. Hence

$$x \in \bigcap_{y \in \sigma} \overline{\mathcal{B}}\left(y, 2t\right).$$

The intersection being non-empty, we deduce $\sigma \in \check{\mathrm{C}}\mathrm{ech}^{2t}(X)$.

Conclusão

Question 1: How to select a t such that $X^t \approx \mathcal{M}$?

Theorem (Frédéric Chazal, David Cohen-Steiner, and André Lieutier, 2009):
Let X and $\mathcal{M}$ be subsets of $\mathbb{R}^n$. Suppose that $\mathcal{M}$ has positive reach, and that $\mathrm{d_H}\left(X, \mathcal{M}\right) \leq \frac{1}{17}\mathrm{reach}\left(\mathcal{M}\right)$.
Then X^t and $\mathcal{M}$ are homotopic equivalent, provided that

$$t \in \left[4\mathrm{d_H}\left(X, \mathcal{M}\right), \mathrm{reach}\left(\mathcal{M}\right) - 3\mathrm{d_H}\left(X, \mathcal{M}\right)\right).$$

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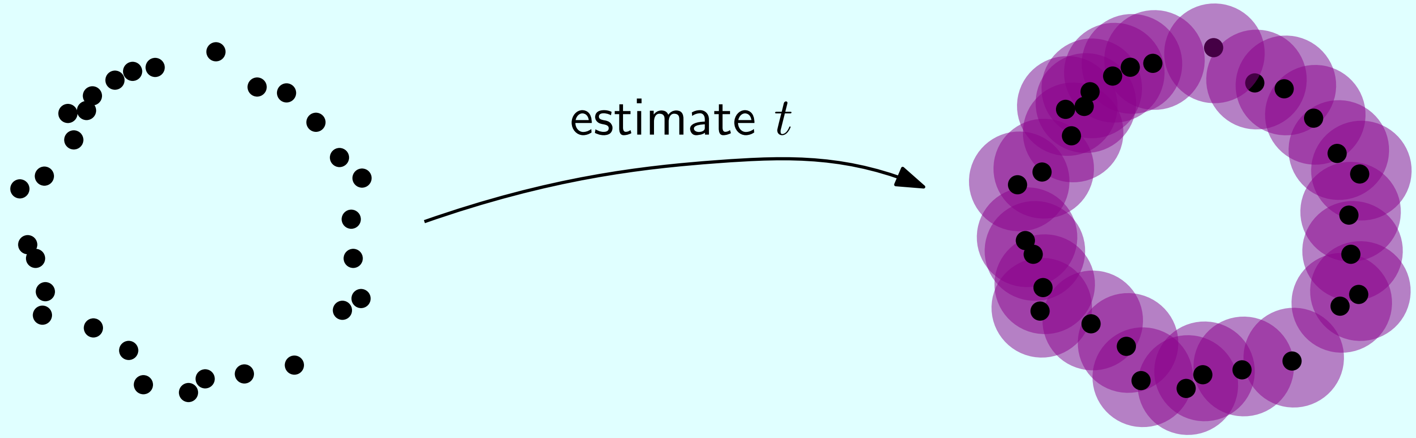

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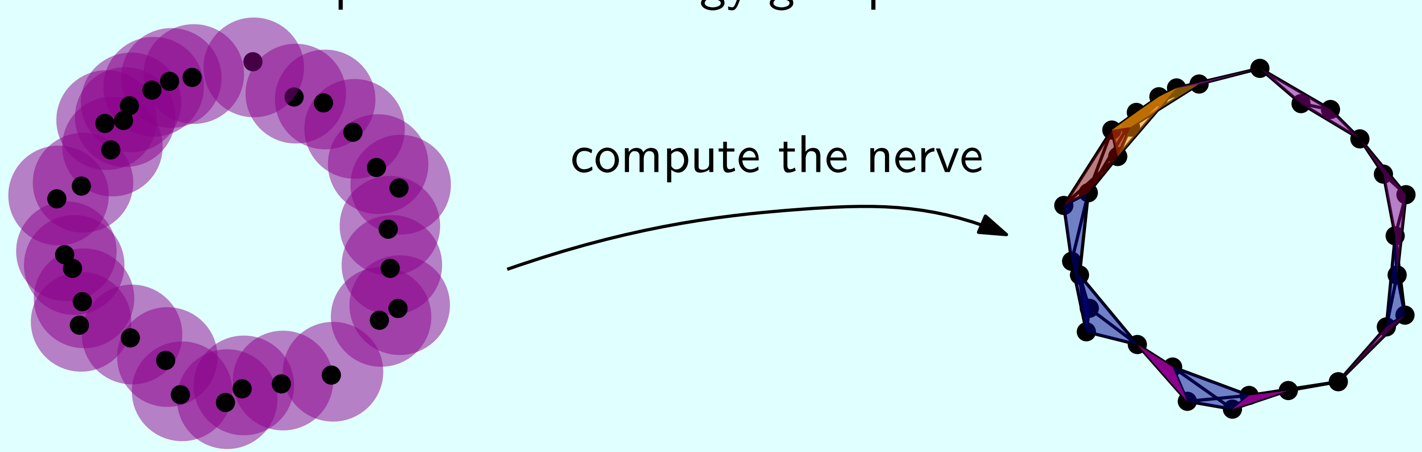

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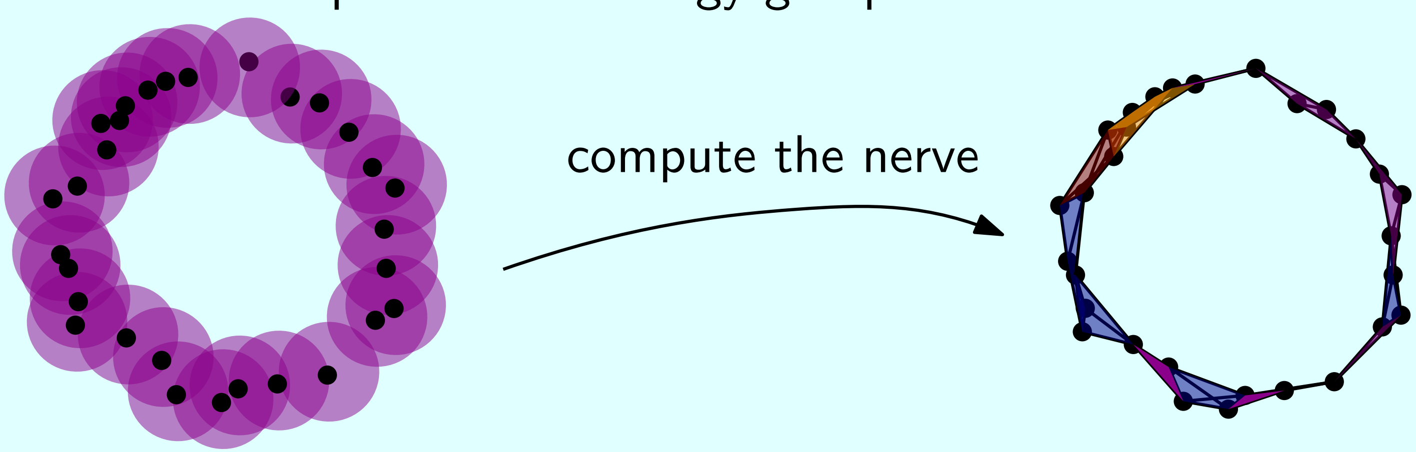

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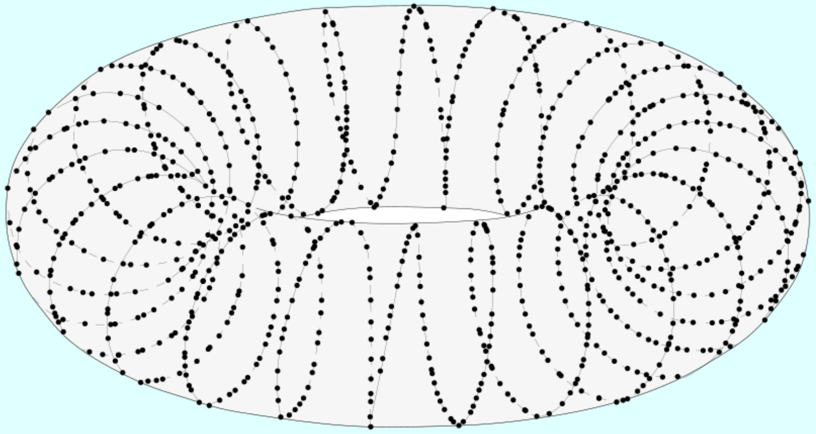
Is this object 1- or 2-dimensional?



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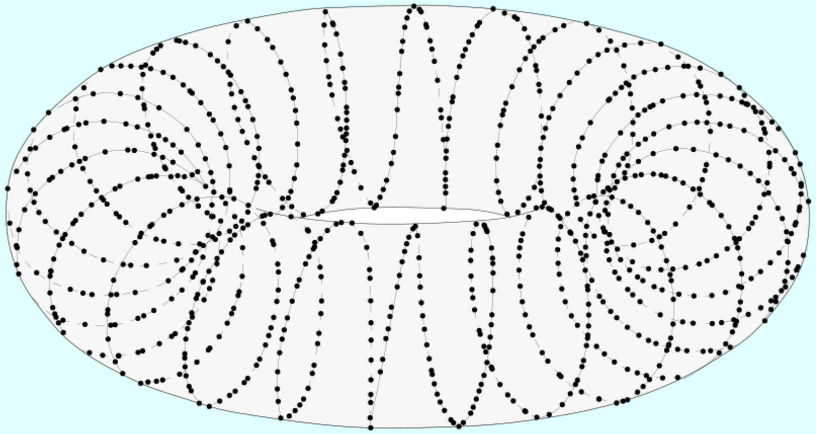

Idea (multiscale analysis): Instead of estimating a value of t, we will choose all of them.