Dynamical Systems & Applications — 26/09/23

An introduction to Topological Data Analysis

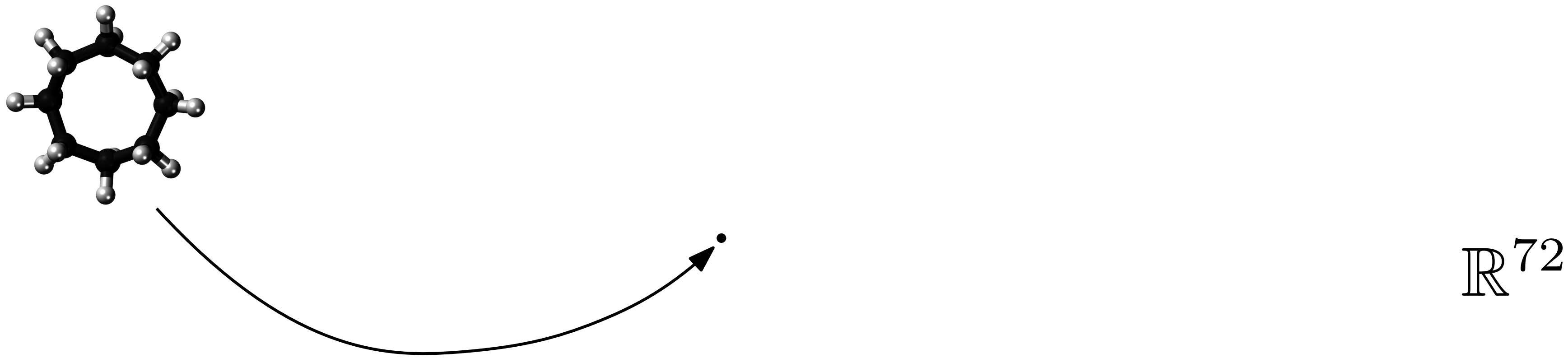
Part I/IV: Topological invariants

https://raphaeltinarrage.github.io

Last update: September 26, 2023

[Martin, Thompson, Coutsias and Watson, Topology of cyclo-octane energy landscape, 2010]

The cyclo-octane molecule $\mathrm{C_8H_{16}}$ contains 24 atoms.
Each atom has 3 spatial coordinates.
Hence a conformation of a molecule can be summarized by **a point** in $\mathbb{R}^{72}$ $(3 \times 24 = 72)$.



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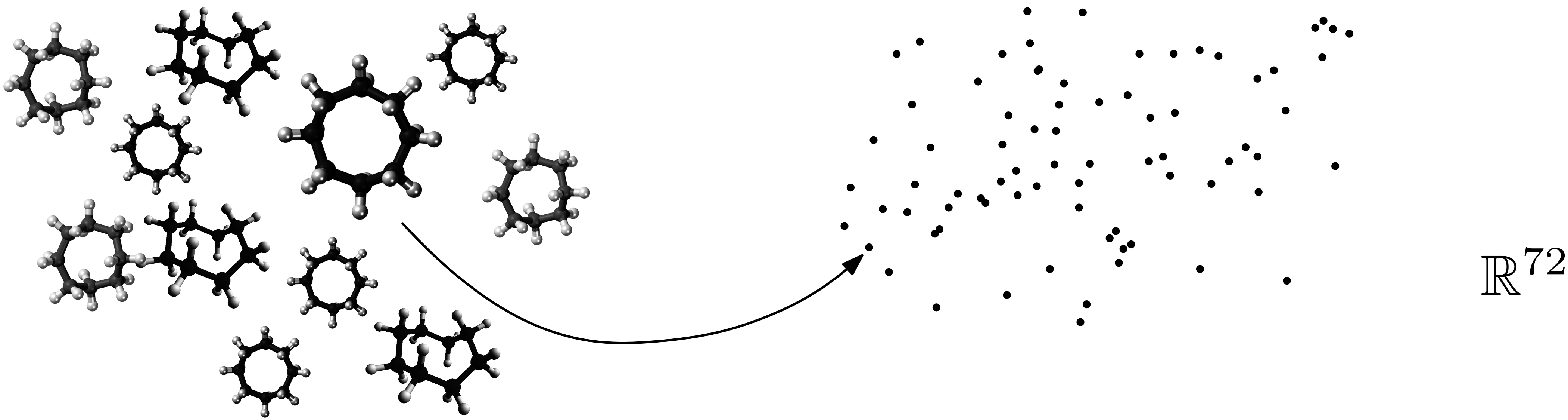

By considering a lot of such molecules, we obtain a **point cloud** in $\mathbb{R}^{72}$.

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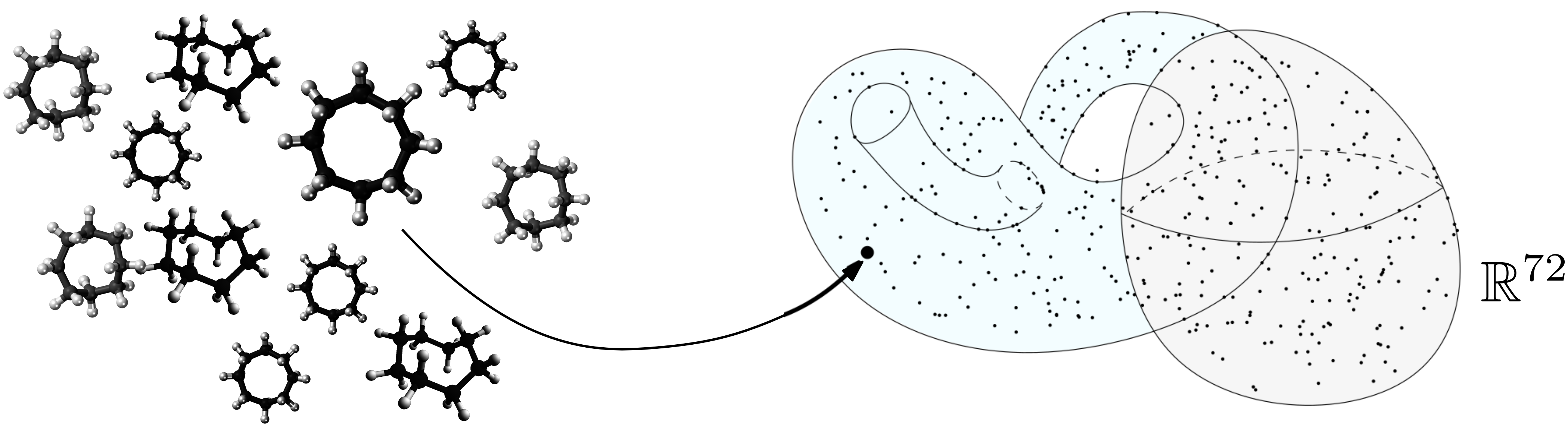
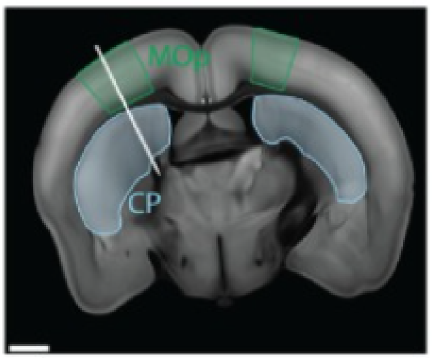


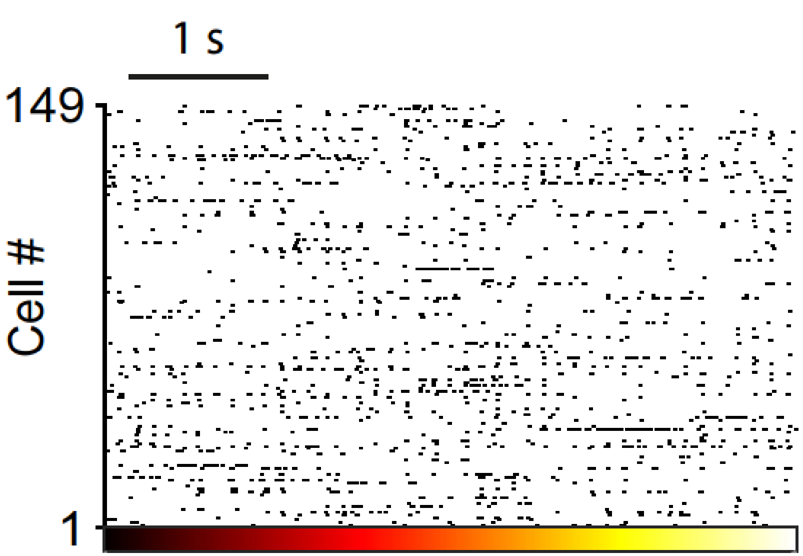
By considering a lot of such molecules, we obtain a **point cloud** in $\mathbb{R}^{72}$.

The authors show that this point cloud lies close to a small dimensional object: **the union of a sphere and a Klein bottle**.

[Richard J. Gardner et al, Toroidal topology of population activity in grid cells, 2022]

The authors recorded spikes of grid cells from rat brains. Then, they applied dimensionality reduction to the firing matrix.





Aperitivo topológico — Neurofisiologia

[Richard J. Gardner et al, Toroidal topology of population activity in grid cells, 2022]

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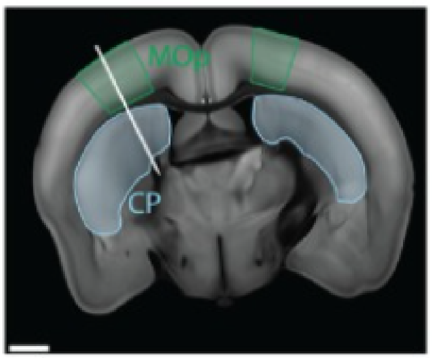
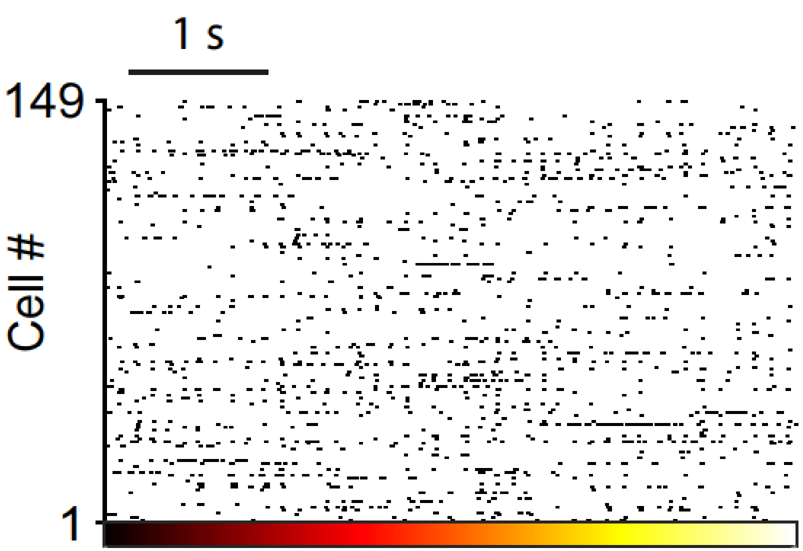
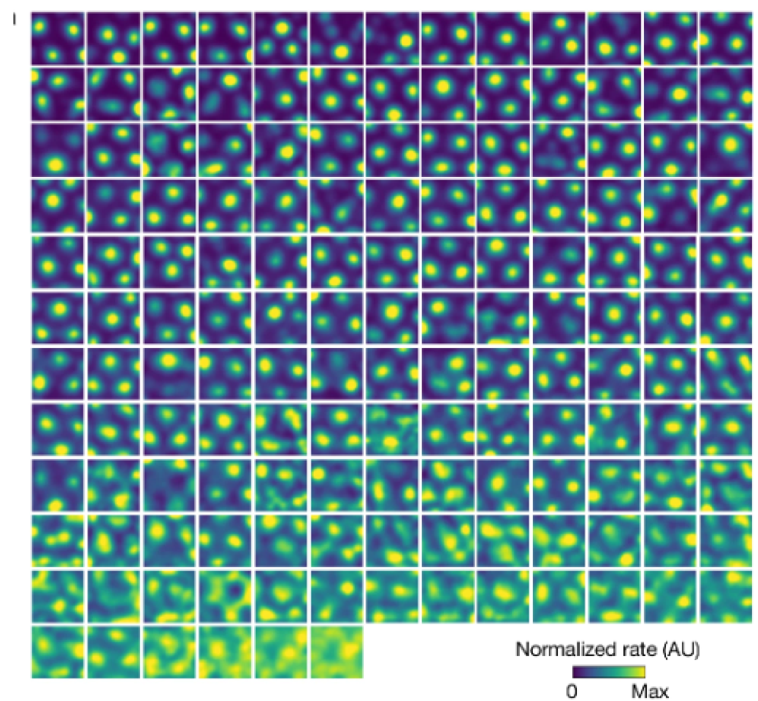
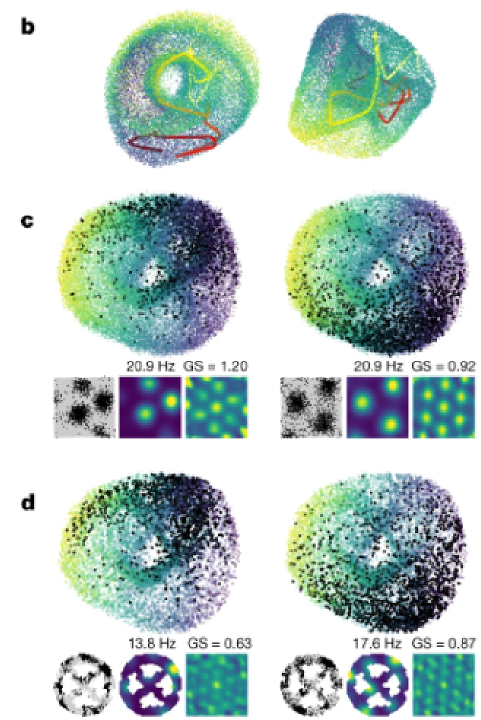
By applying persistent homology, they observed the homology of a **torus**.







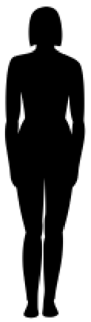


Aperitivo topológico — Biologia

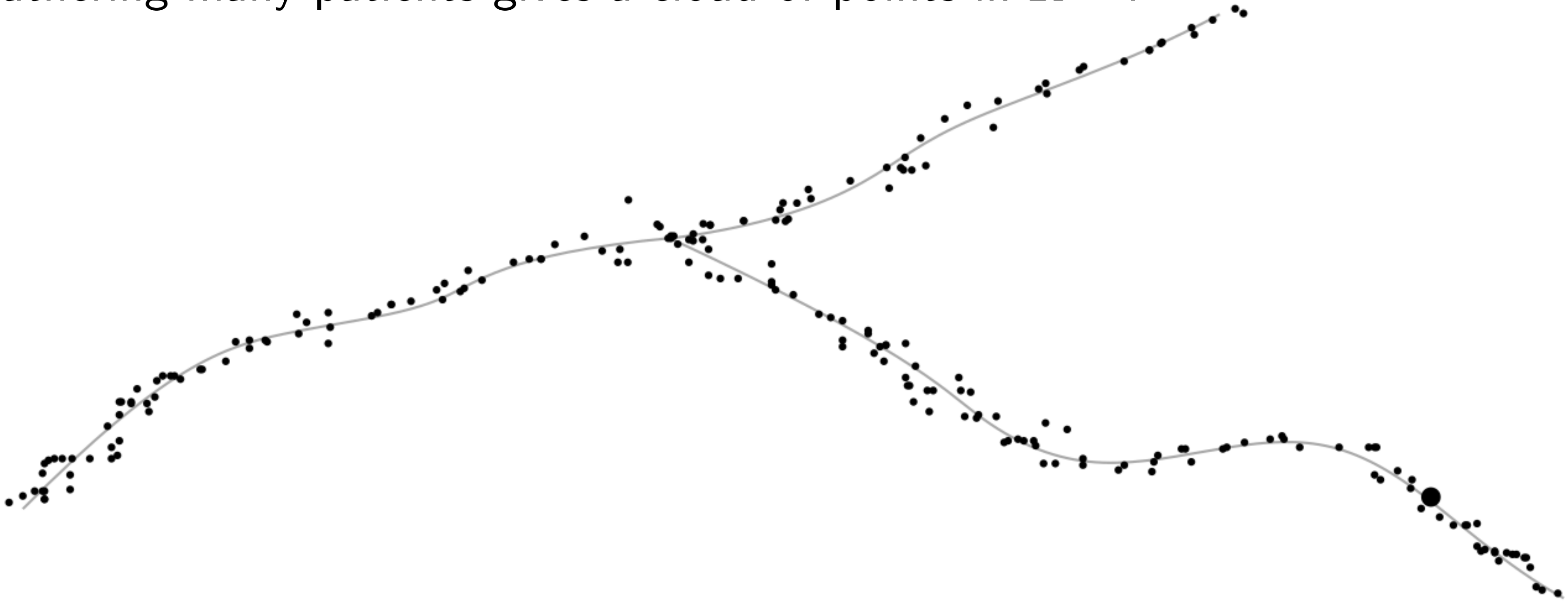
[Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival, Monica Nicolau, Arnold J Levine, and Gunnar Carlsson, *Proceedings of the National Academy of Sciences*, 2011]

The authors study tissues from patients infected by breast cancer. They obtain 262 genomic variables per patient.

$$(x_1, x_2, ..., x_{262}) \longleftarrow$$



Gathering many patients gives a cloud of points in $\mathbb{R}^{262}$.



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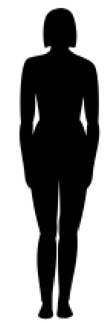

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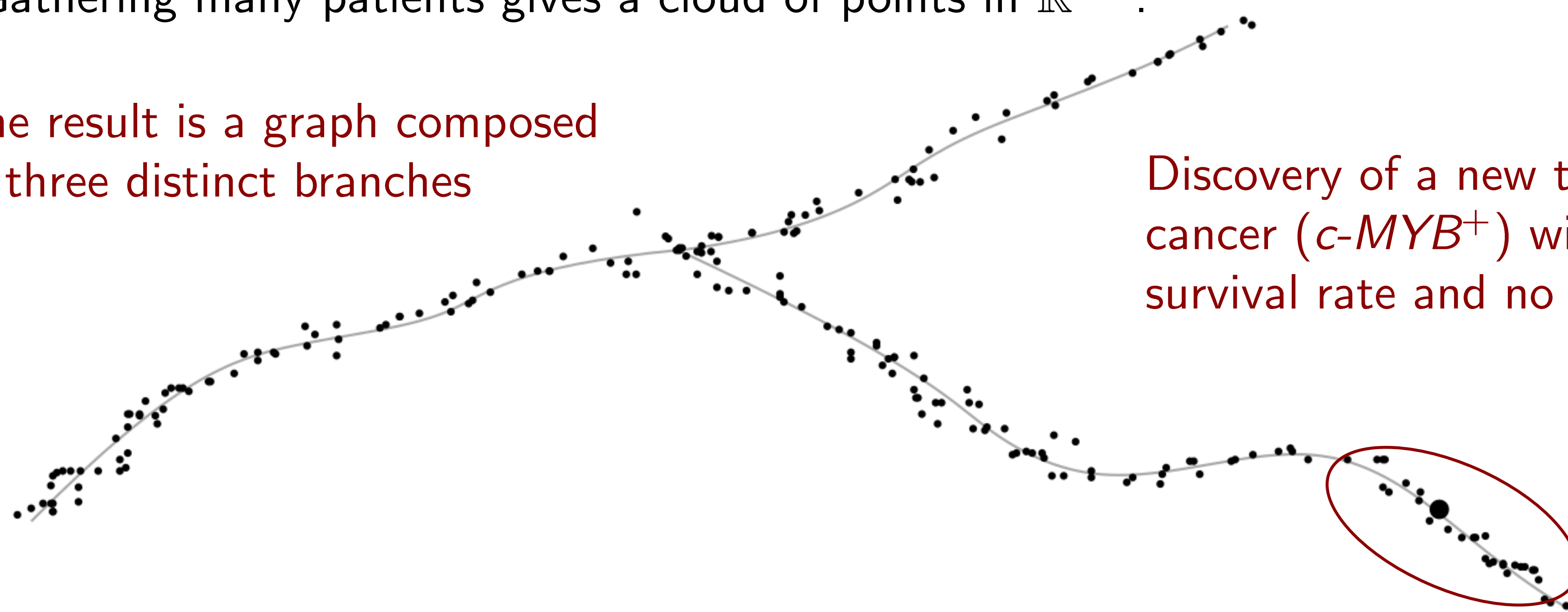
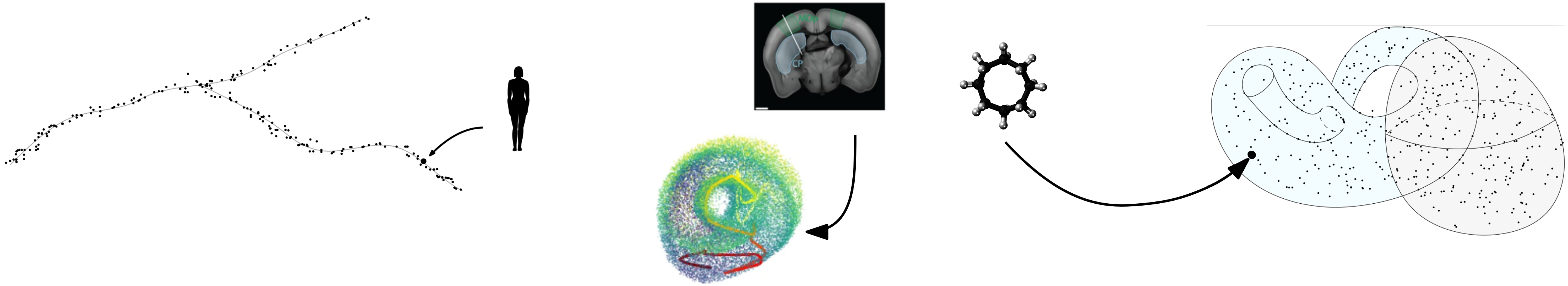
The result is a graph composed of three distinct branches

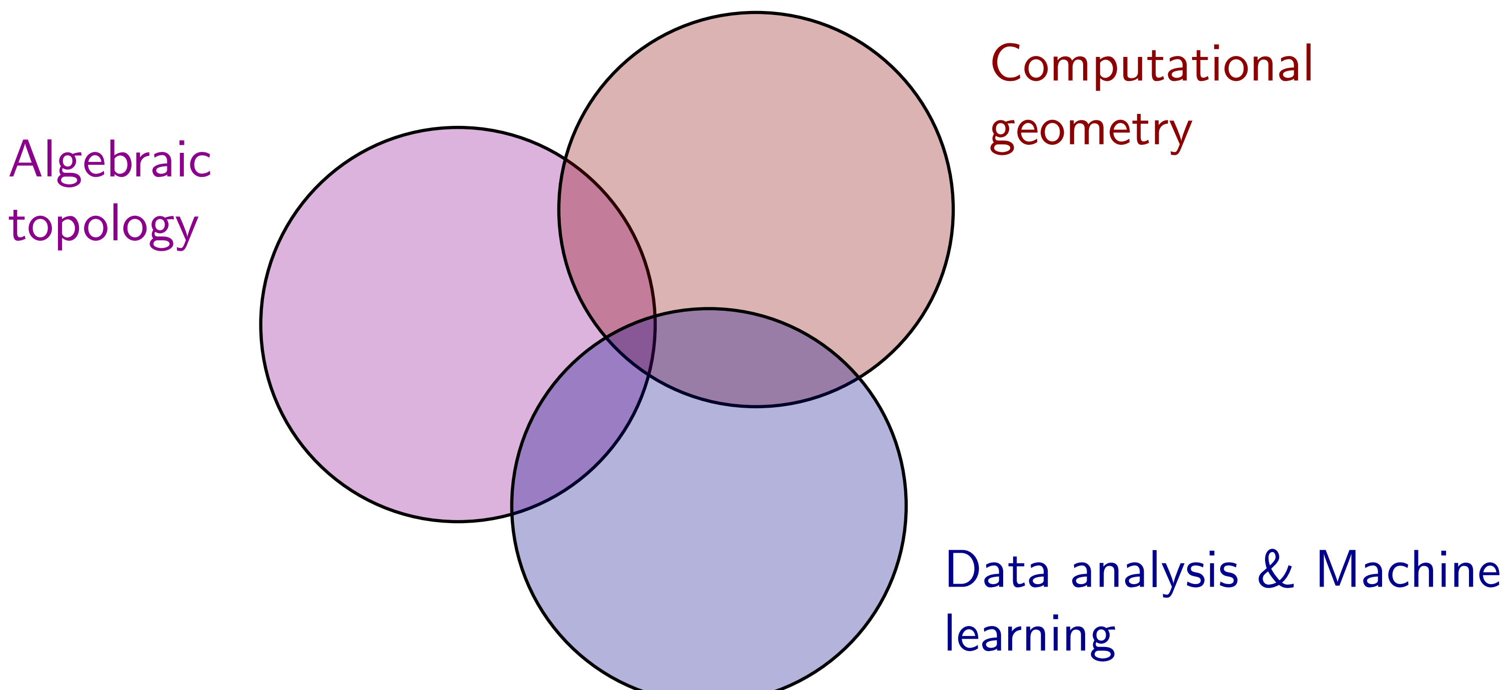


Discovery of a new type of breast cancer (*c-MYB*$^+$) with a 100% survival rate and no metastases

Qual é a forma dos dados?



Topological Data Analysis (TDA) allows to **explore** and **understand** the topology of datasets.



Part I/IV: Topological invariants

Tuesday 26th

Part II/IV: Homology

Thursday 28th

Part III/IV: Persistent Homology

Tuesday 3rd

Part IV/IV: Python tutorial

Thursday 5th

Algumas figuras históricas

Euler

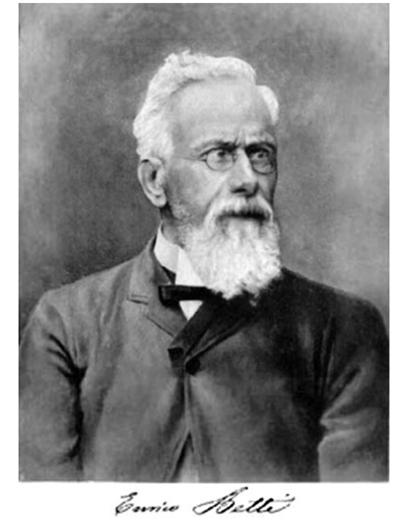
1736: Solutio problematis ad geometriarn situs pertinentis

Möbius

1865: Bestimmung des Inhaltes eines Polyëders

Riemann

1857: Theorie der Abel'schen Functionen



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1871: Sopra gli spazi di un numero qualunque di dimensioni

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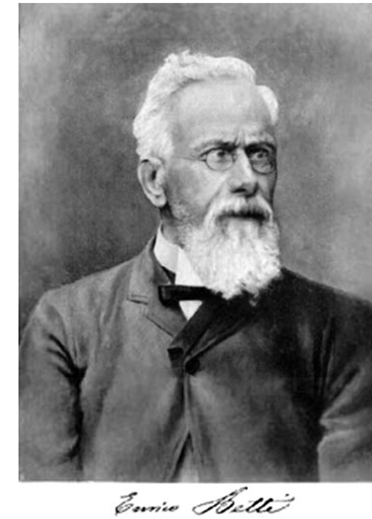

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1895: Analysis Situs

Toute 3-variété compacte sans bord et simplement connexe est-elle homéomorphe à la 3-sphère ?

[Cinquième complément à l'analysis situs, 1904]

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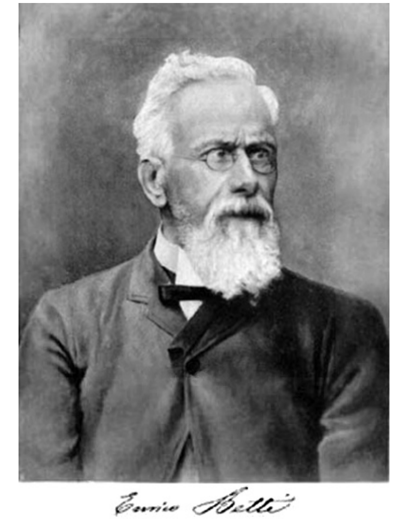

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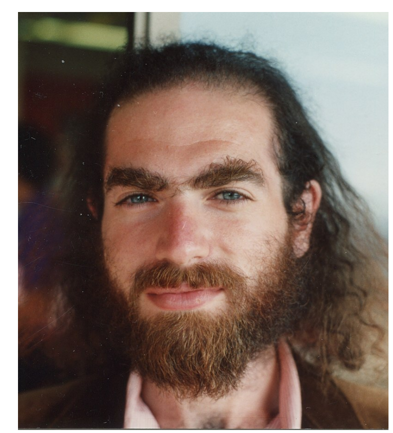

Perelman

2002: The entropy formula for the Ricci flow and its geometric applications

Protagonistas da topologia

In topology we study **topological spaces**.

Definition: a topological space is a set X endowed with a collection of **open sets** $\{O_\alpha \mid \alpha \in A\}$, with $O_\alpha \subset X$, such that

- $\emptyset$ and X are open sets,
- an infinite union of open sets is an open set,
- a finite intersection of open sets is an open set.

Definition: Given two topological spaces X and Y, a map $f\colon X \to Y$ is **continuous** if for every open set $O \subset Y$, the preimage $f^{-1}(O)$ is an open set of X.

$$X \longrightarrow Y$$

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translation in
ϵ-δ calculus

One can think of **subsets** $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$,

and maps $f\colon X \to Y$ **continuous** in the following sense:

$$\forall x \in X, \forall \epsilon > 0, \exists \eta > 0, \forall y \in X, \|x - y\| < \eta \implies \|f(x) - f(y)\| < \epsilon.$$

1 - We can build a topological space by seeing it as a **subspace** of another one.

In $\mathbb{R}^n$, we can define:

- the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$
- the unit cube $\mathcal{C}_{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid \max(|x_1|, ..., |x_n|) = 1\}$
- the open balls $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$
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Construção de espaços topológicos

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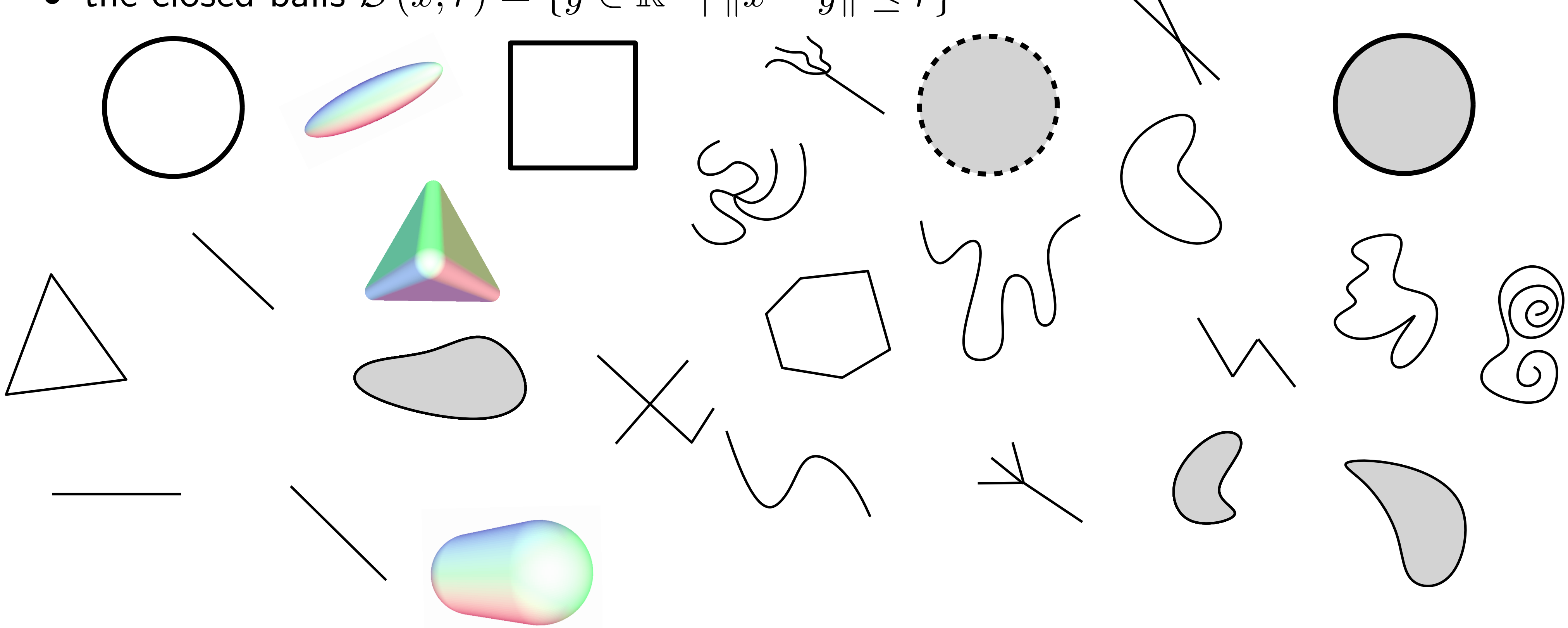

Most of the time, we do not have a nice algebraic definition...

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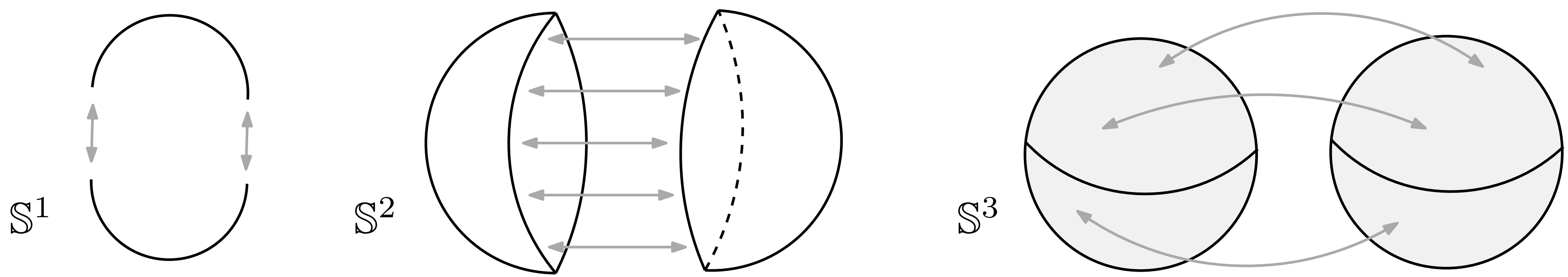

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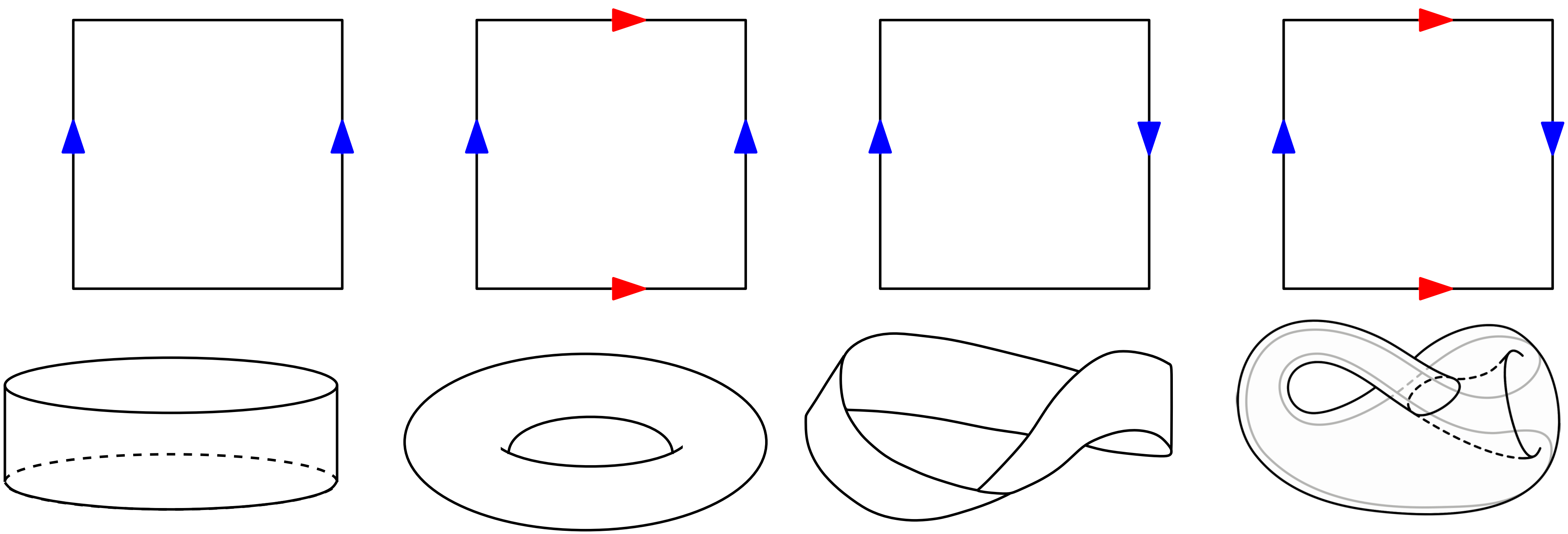
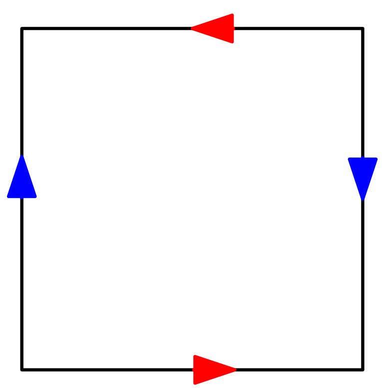


Bonus:



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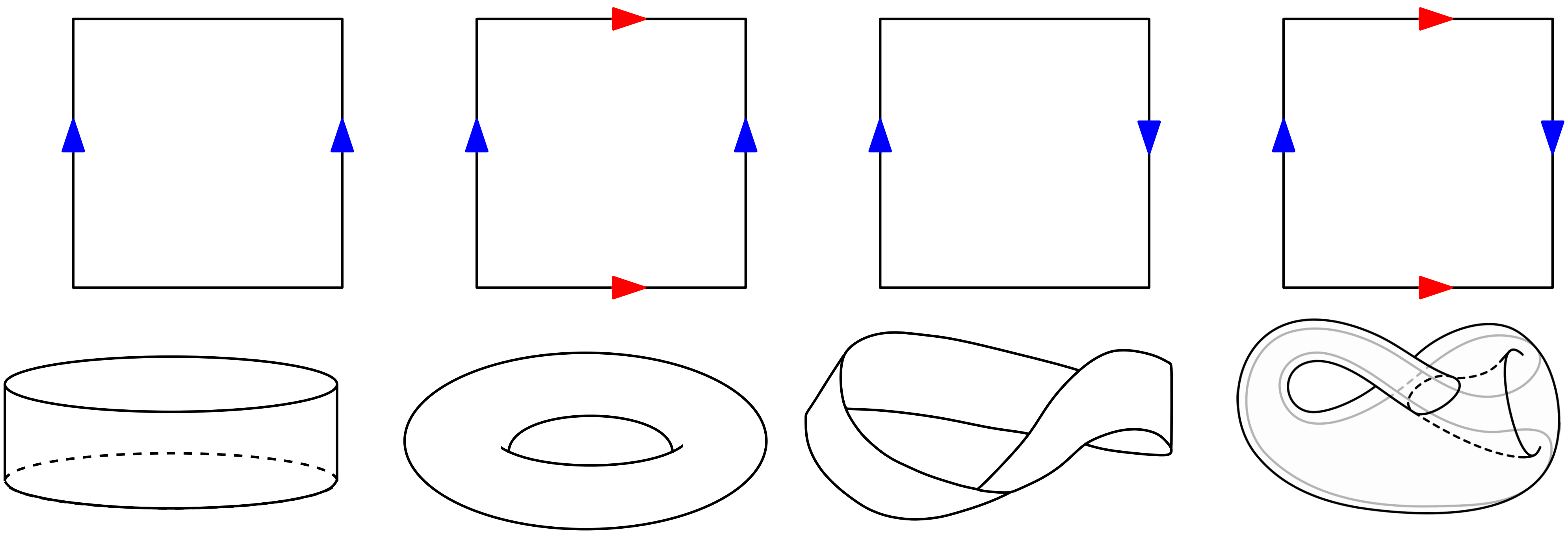

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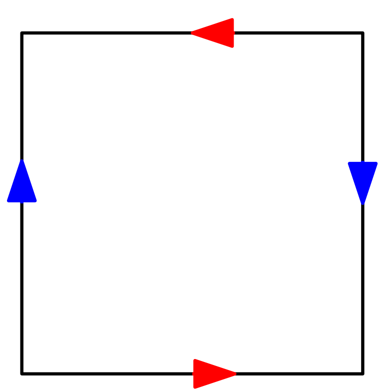

Projective plane $\mathbb{R}P^2$

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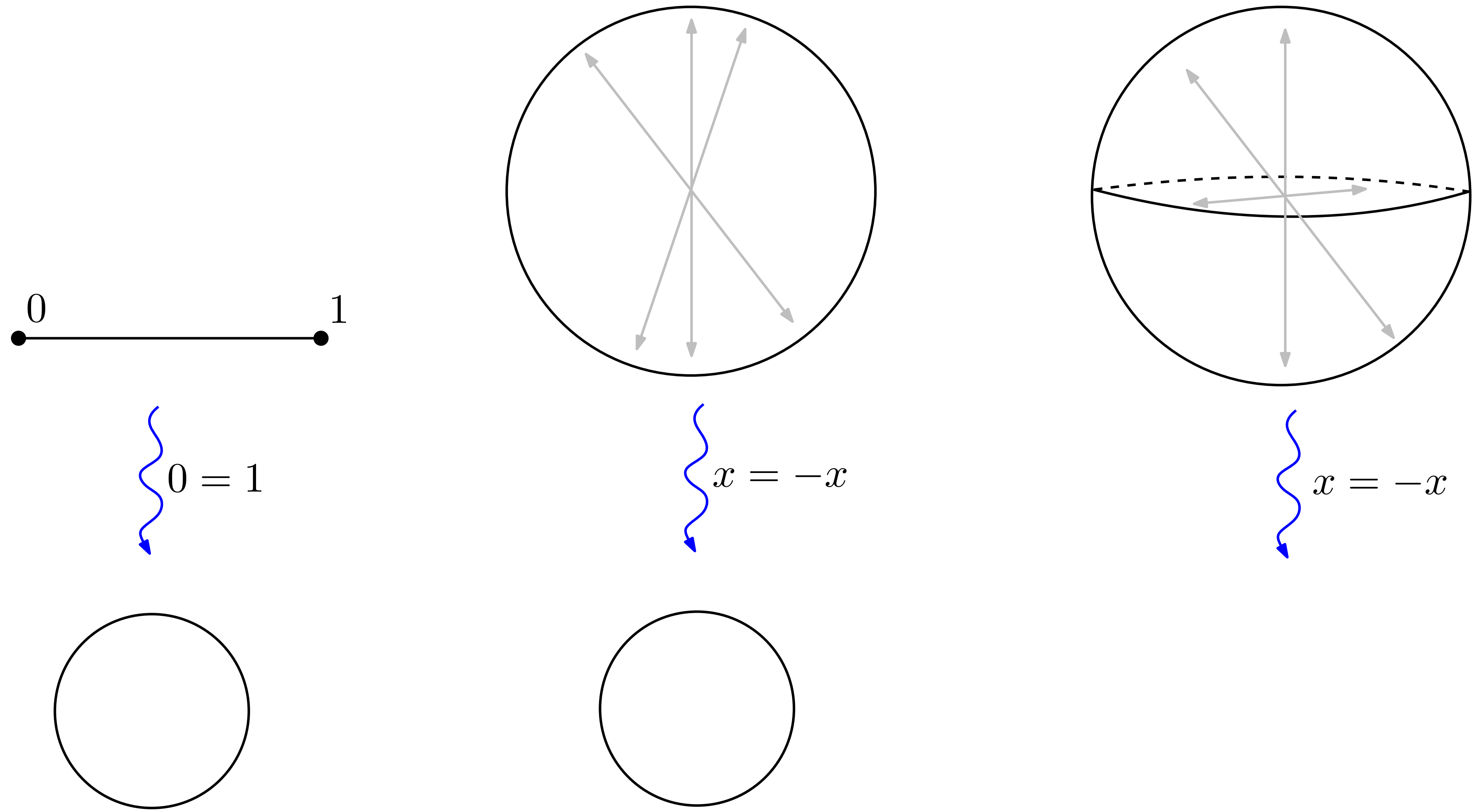

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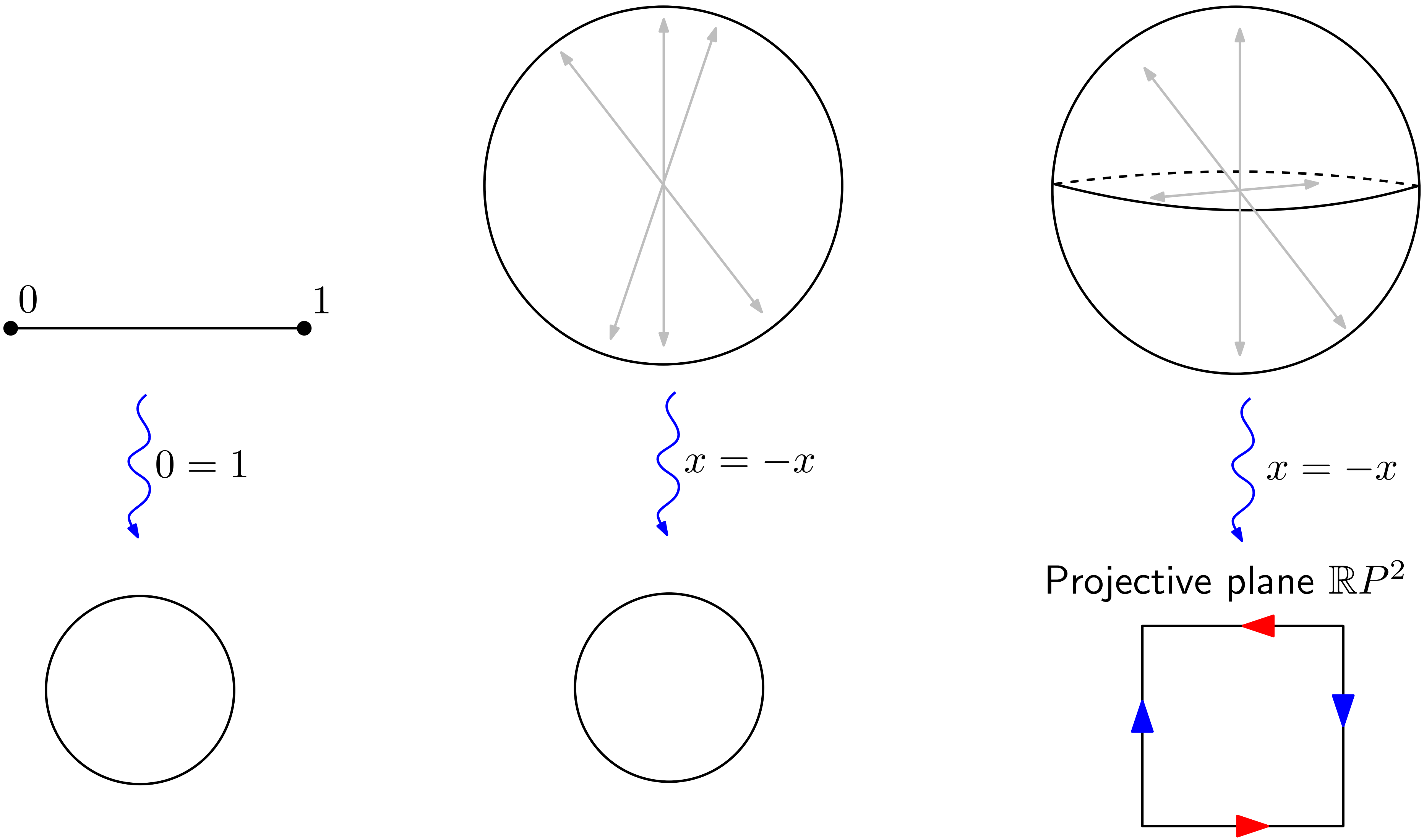

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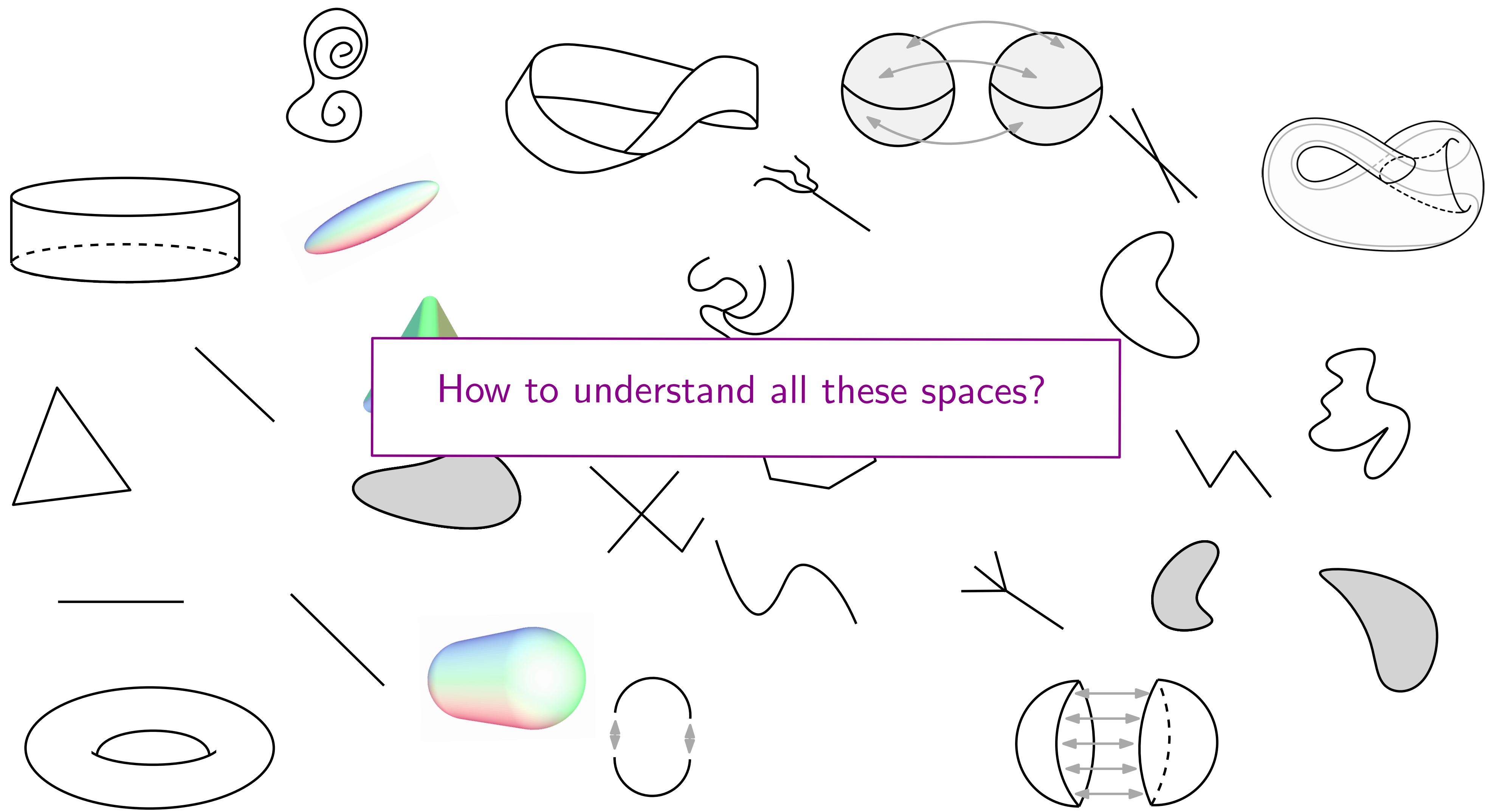

How to understand all these spaces?

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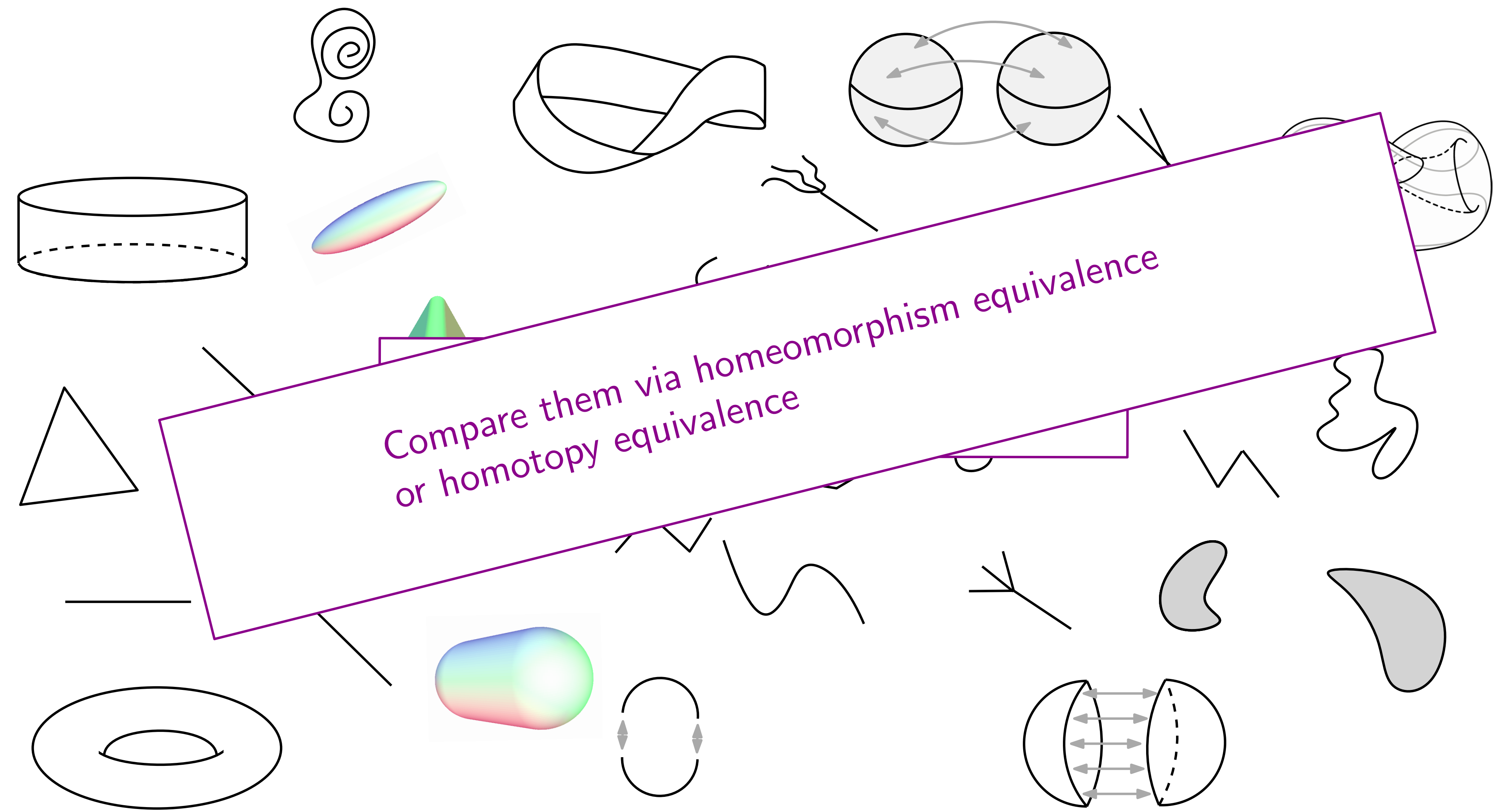

Homeomorfismos

Definition: Let X and Y be two topological spaces, and $f\colon X \to Y$ a map.
We say that f is a **homeomorphism** if

- f is a bijection,
- $f\colon X \to Y$ is continuous,
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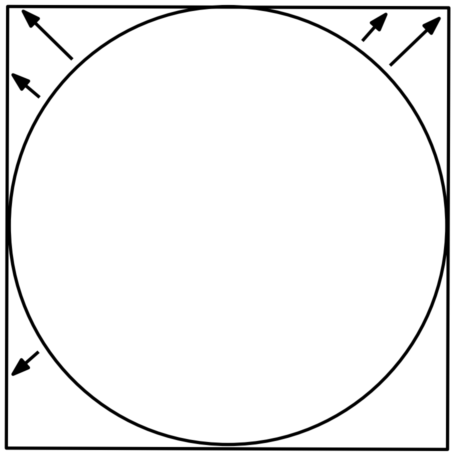
If there exist such a homeomorphism, we say that the two topological spaces are **homeomorphic**.

Example: The unit circle and the unit square are homeomorphic via

$$\begin{aligned} f\colon \mathbb{S}_1 &\longrightarrow \mathcal{C} \\ (x_1, x_2) &\longmapsto \frac{1}{\max(|x_1|, |x_2|)}(x_1, x_2) \end{aligned}$$



Interpretation: Homeomorphisms allow 'continuous deformations'.

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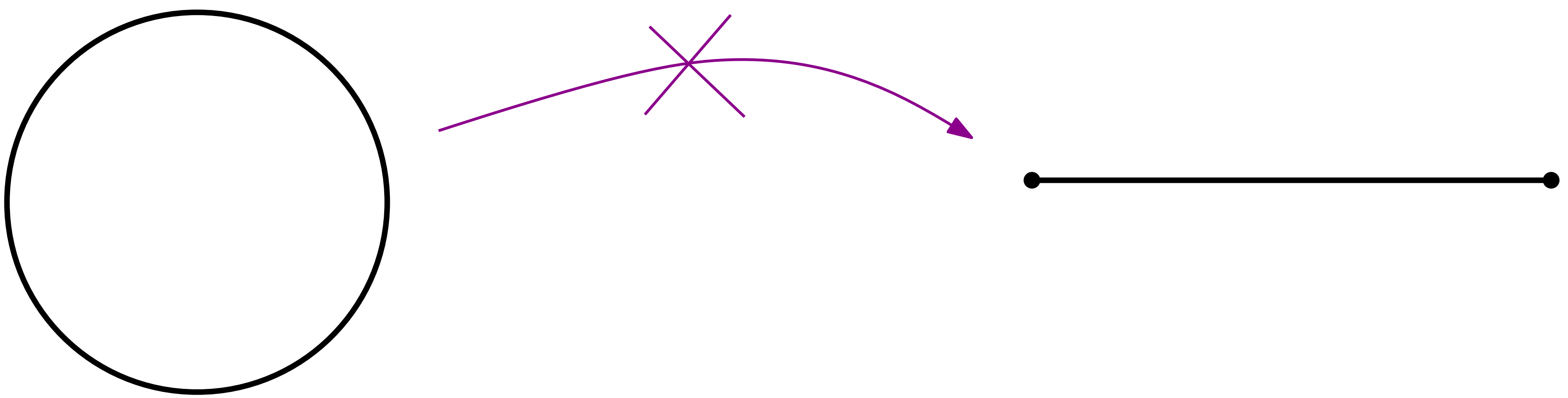
Example: The unit circle and the interval $[0, 1]$ are not homeomorphic.



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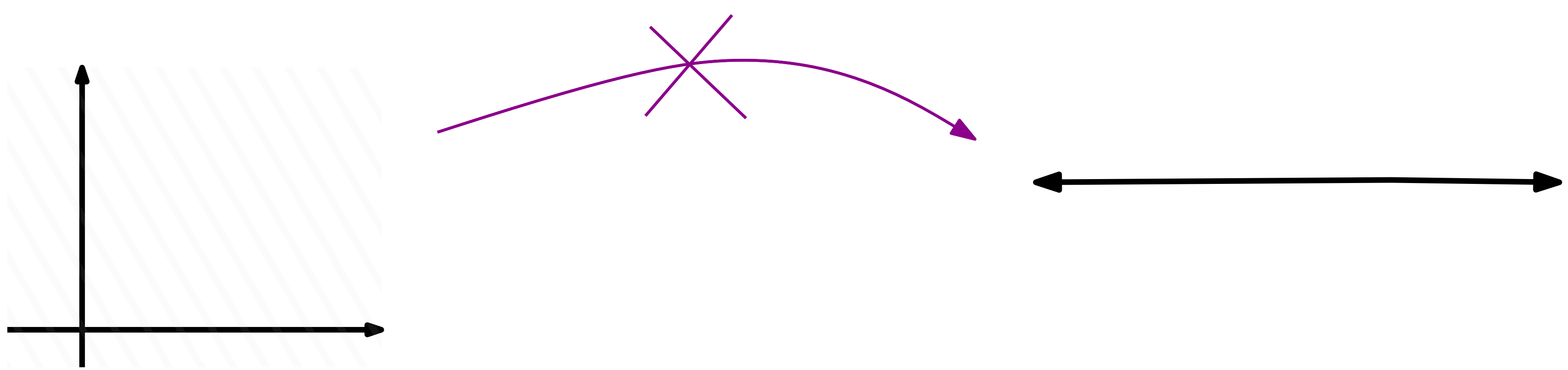
Example (Invariance of domain): [Brouwer, 1912] If $n \neq m$, the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.



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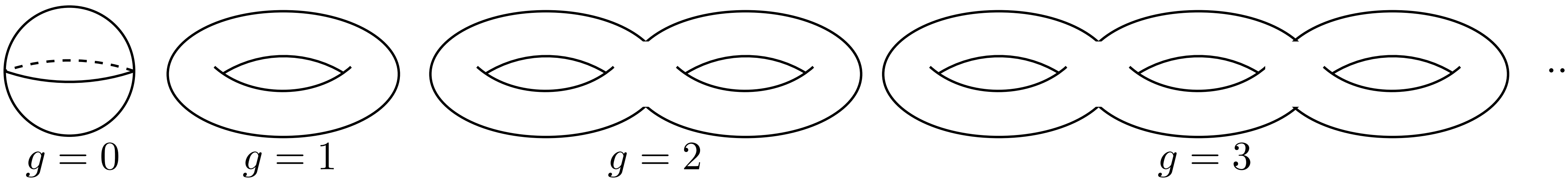
Example (Classification of surfaces): [Möbius, Jordan, von Dyck, Dehn and Heegaard, Alexander, Brahana, 1863-1921] If $g \neq g'$, the surfaces of genus g and g' are not homeomorphic.



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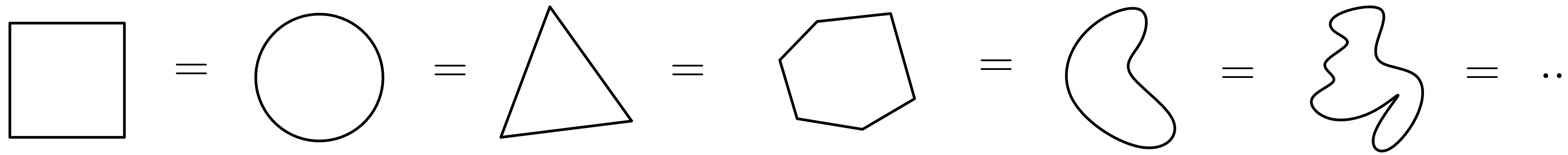
Classes de homeomorfismo

We can gather topological spaces that are homeomorphic



the class of circles

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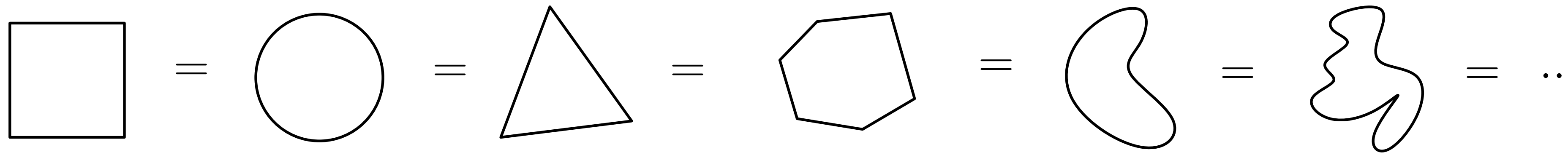

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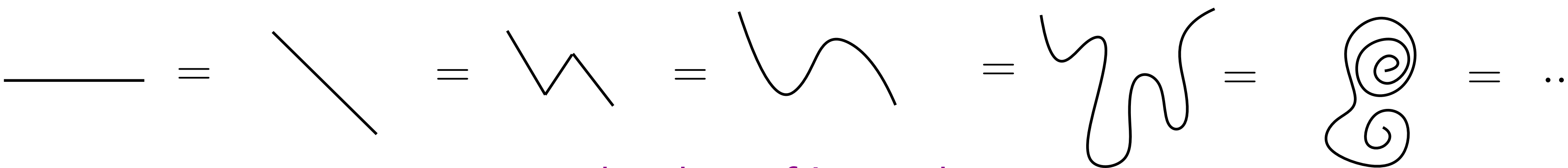

the class of intervals

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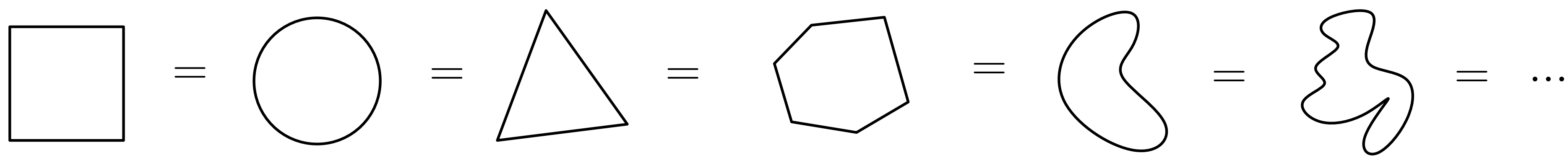

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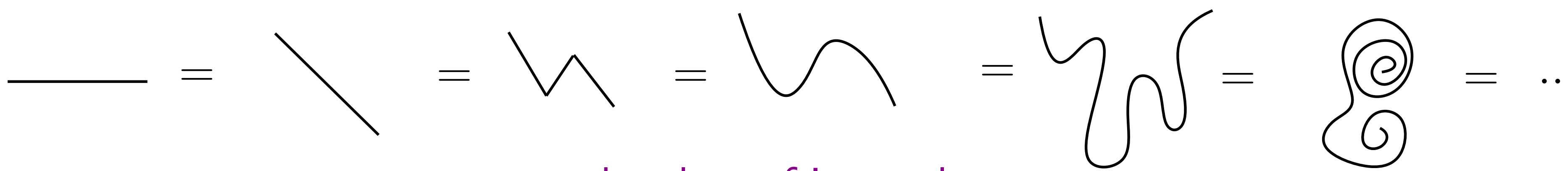

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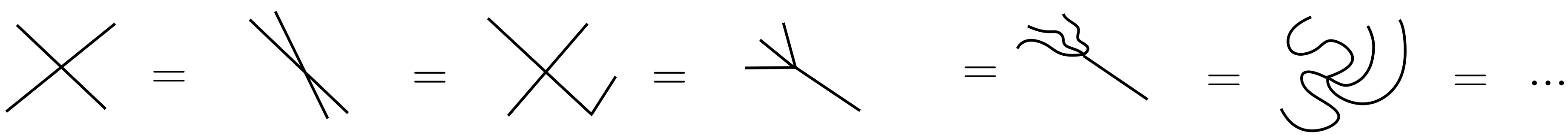

the class of crosses

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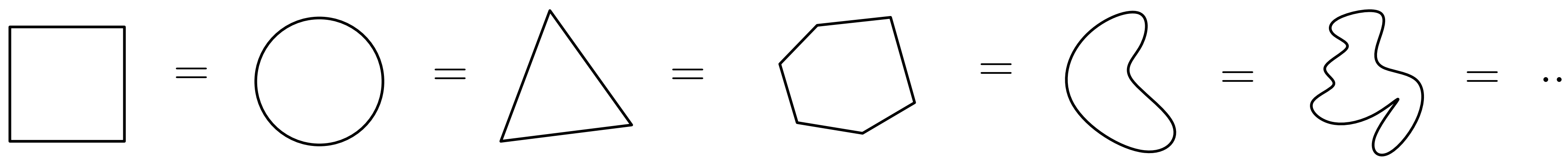

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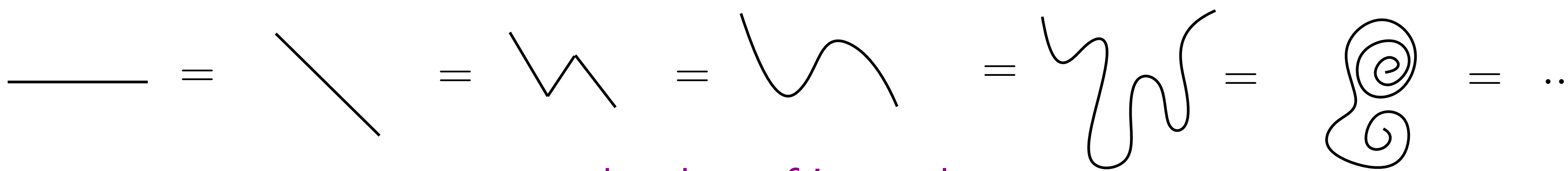

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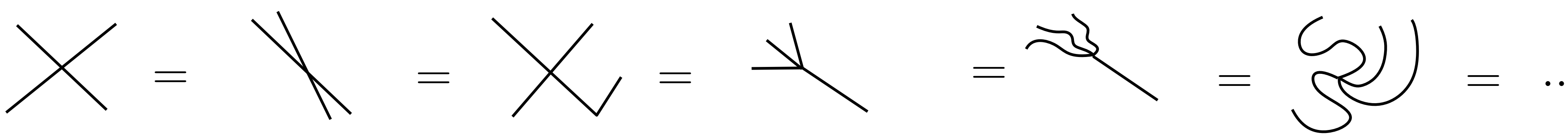

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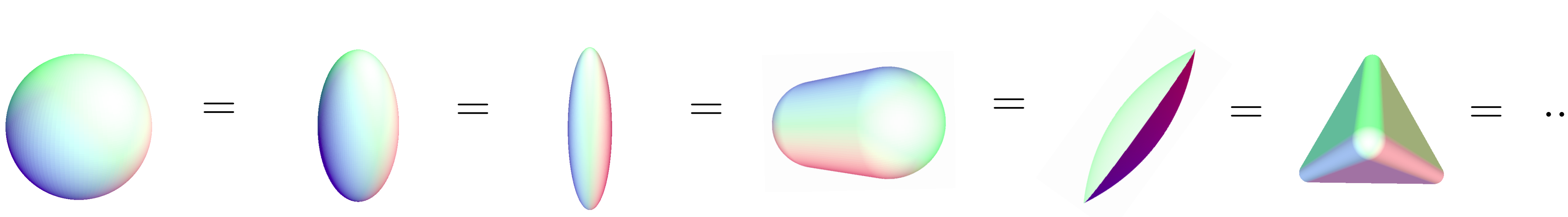
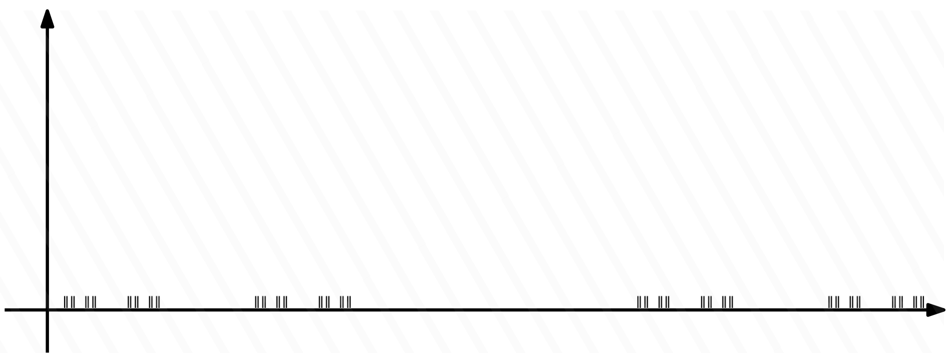


the class of spheres

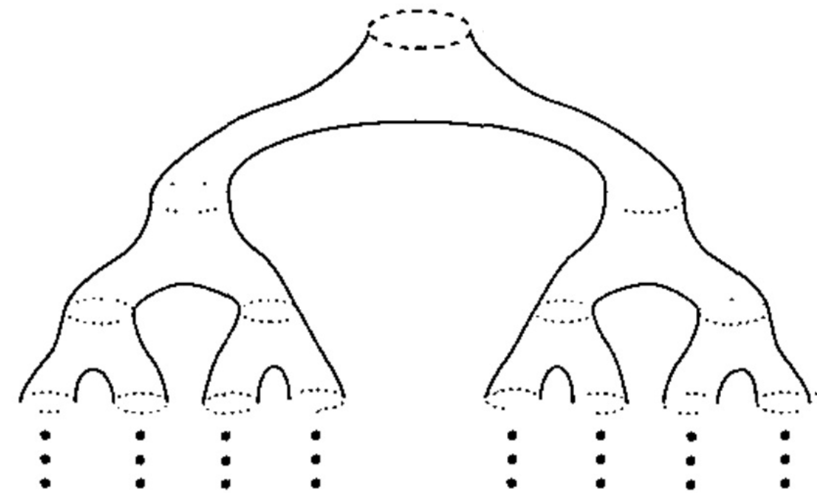
Classes de homeomorfismo

In general, it may be complicated to determine whether two spaces are homeomorphic.



$\mathbb{R}^2\setminus$ Cantor set

$$\stackrel{?}{=}$$



To answer this problem, we will use the notion of **invariant**.

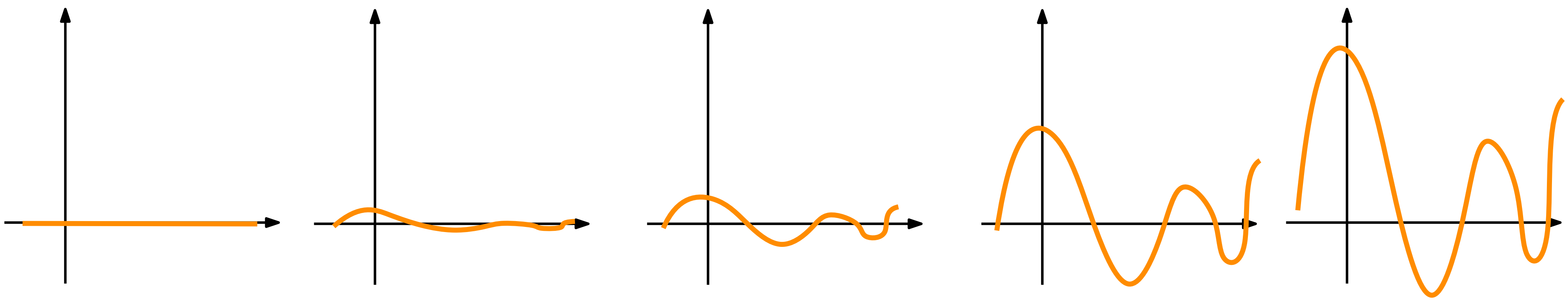
Homotopias

Definition: Let X, Y be two topological spaces, and $f, g\colon X \to Y$ two continuous maps. A **homotopy** between f and g is a map $F\colon X \times [0,1] \to Y$ such that:

- $x \mapsto F(x,0)$ is equal to f,
- $x \mapsto F(x,1)$ is equal to g,
- $F\colon X \times [0,1] \to Y$ is continuous.

If such a homotopy exists, we say that the maps f and g are **homotopic**.

Example: Homotopy between $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$.



$F(\cdot,0) = f$ $\quad$ $F(\cdot,0.2)$ $\quad$ $F(\cdot,0.5)$ $\quad$ $F(\cdot,0.6)$ $\quad$ $F(\cdot,1) = g$

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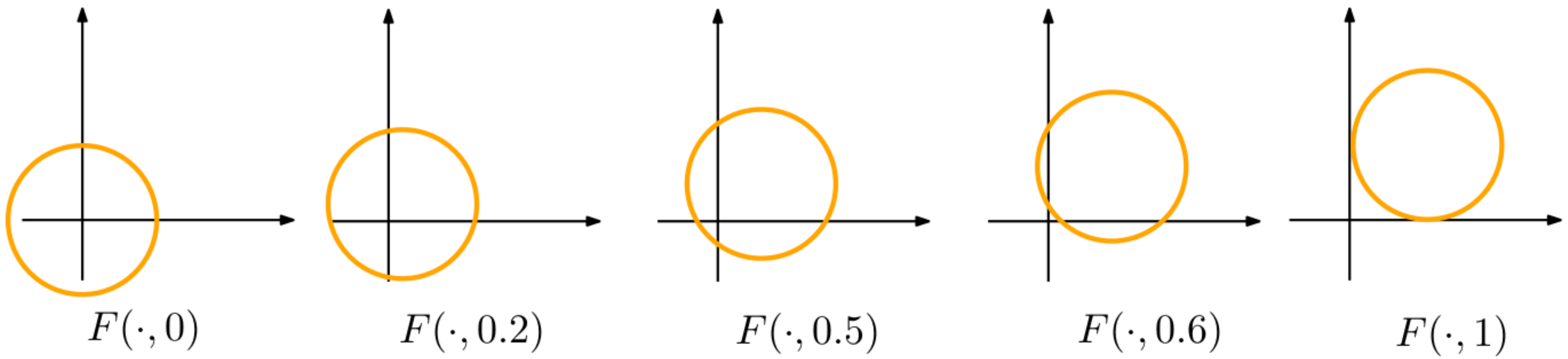
Example: The map $F\colon (x,t) \in \mathbb{S}^1 \times [0,1] \mapsto (\cos(\theta) + 2t, \sin(\theta) + 2t)$ is a homotopy between

$$\begin{aligned} f\colon \mathbb{S}^1 &\to \mathbb{R}^2 \\ \theta &\mapsto (\cos(\theta), \sin(\theta)) \end{aligned} \quad \text{and} \quad \begin{aligned} g\colon \mathbb{S}^1 &\to \mathbb{R}^2 \\ \theta &\mapsto (\cos(\theta)+2, \sin(\theta)+2) \end{aligned}$$



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not well-defined

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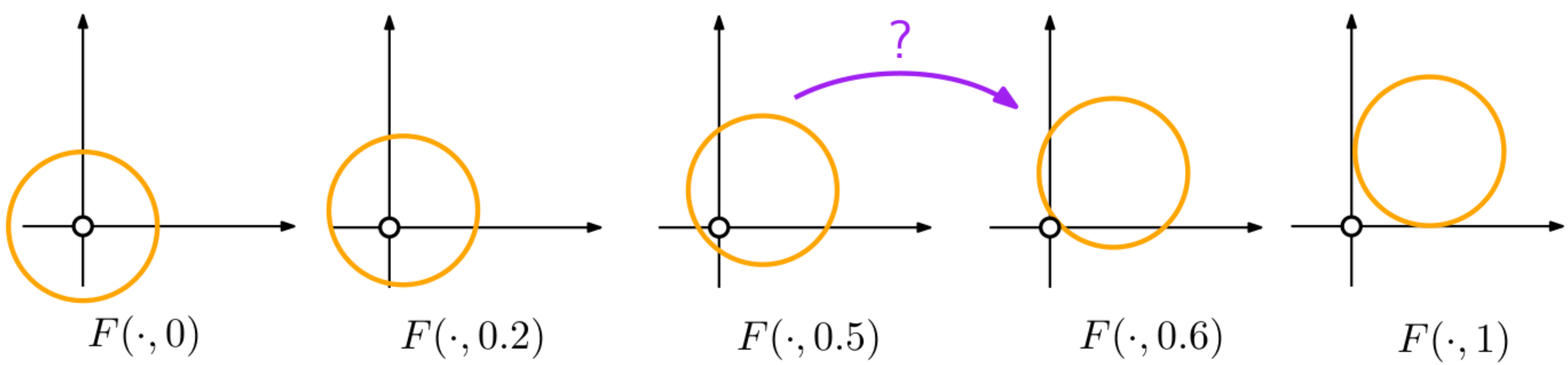
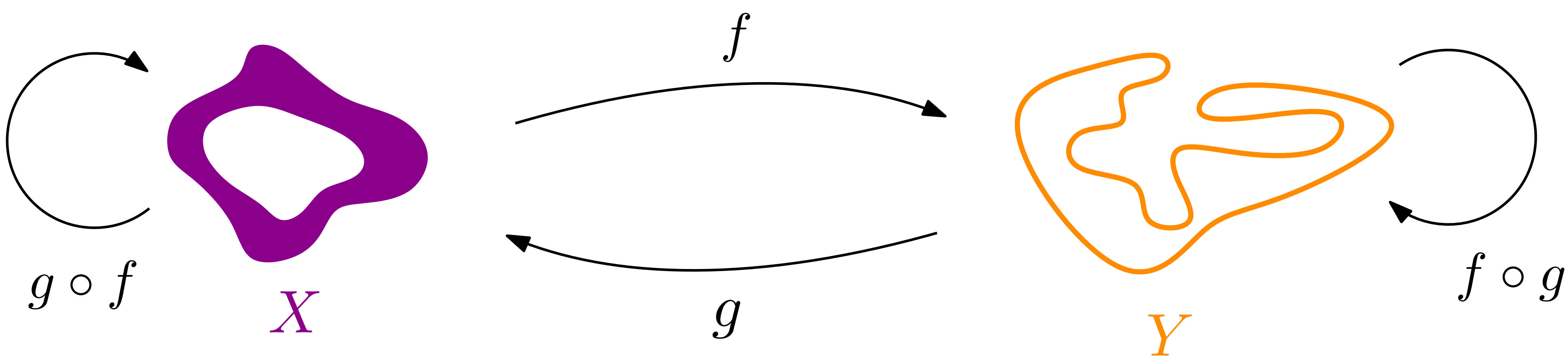


This is not true anymore if we remove the origin from the plane.

Defintion: Let X and Y be two topological spaces. A **homotopy equivalence** between X and Y is a pair of continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that:

- $g \circ f\colon X \to X$ is homotopic to the identity map $\mathrm{id}\colon X \to X$,
- $f \circ g\colon Y \to Y$ is homotopic to the identity map $\mathrm{id}\colon Y \to Y$.

If such a homotopy equivalence exists, we say that X and Y are **homotopy equivalent**.



Equivalência de homotopia

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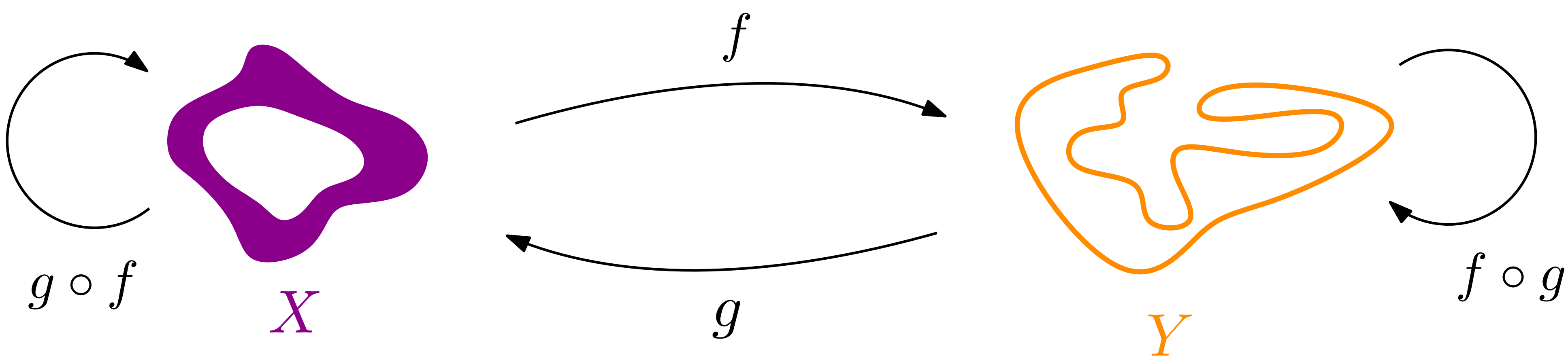

Observation: A homotopy equivalence is a weaker formulation of homeomorphism.

$$g \circ f = \mathrm{id}_X \qquad X \underset{g = f^{-1}}{\overset{f}{\rightleftarrows}} Y \qquad f \circ g = \mathrm{id}_Y$$

Equivalência de homotopia

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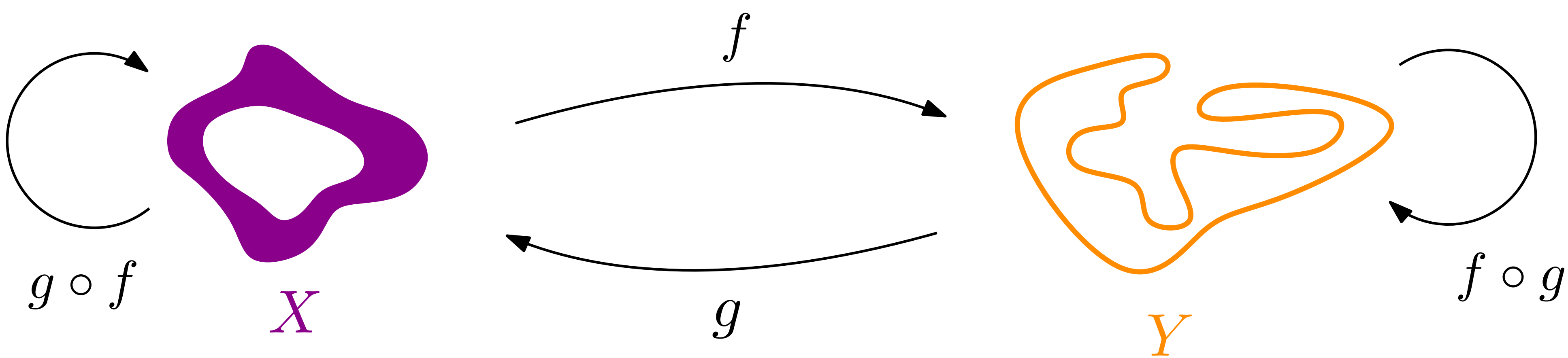

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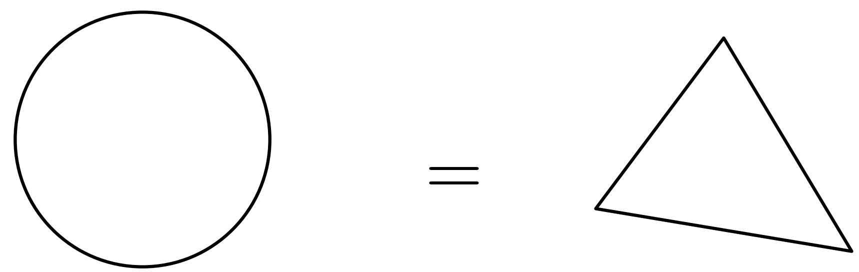
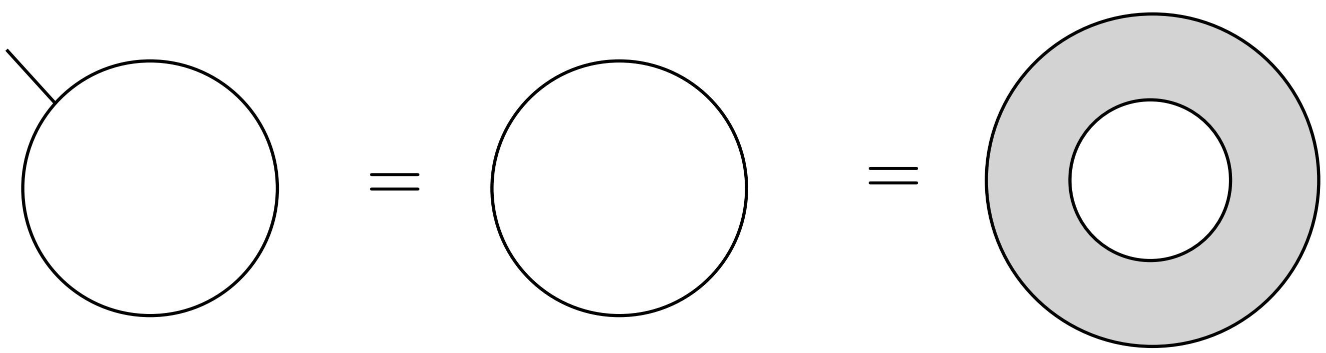
Proposition: If two topological spaces are homeomorphic, then they are homotopy equivalent.

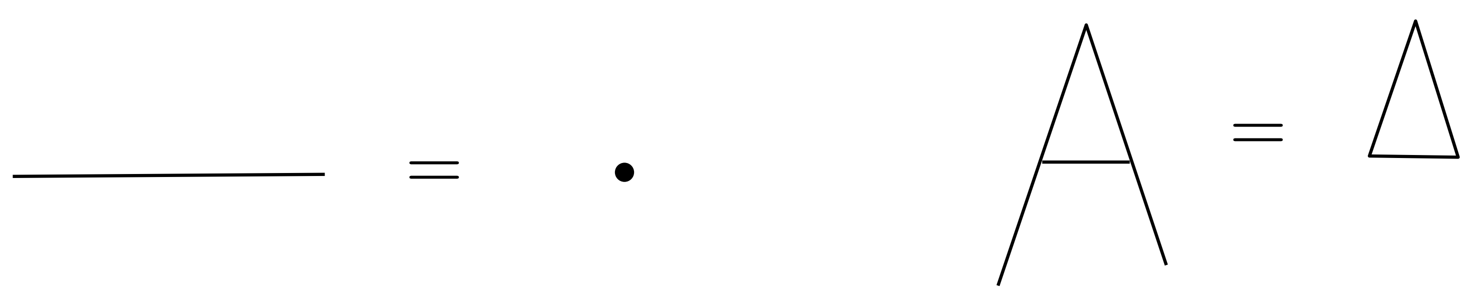
Equivalência de homotopia

Homotopy equivalence allows to continuously **deform** the space



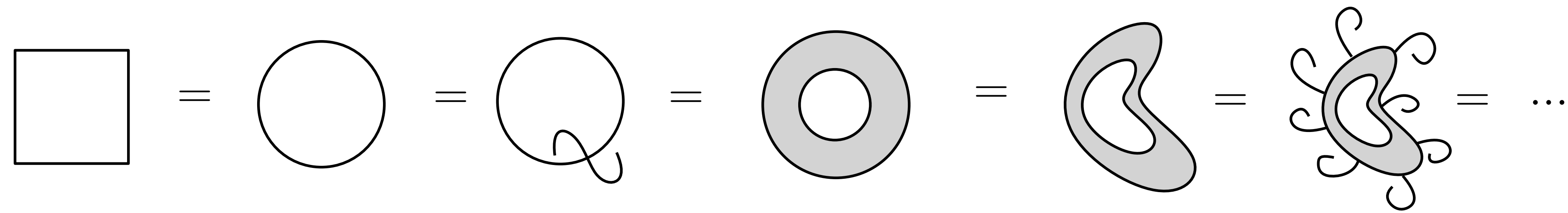
and to **retract** it.



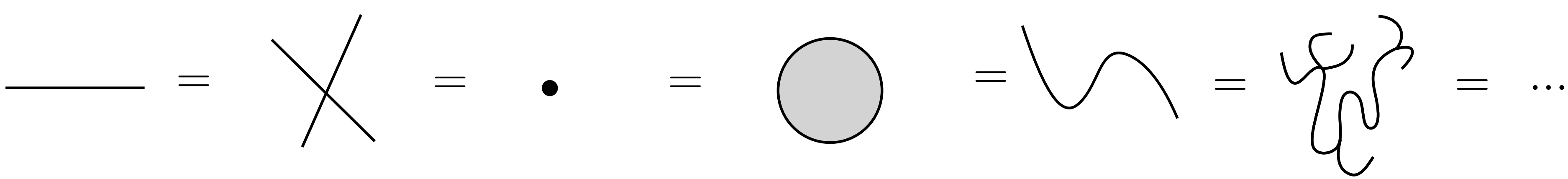


Classes de homotopia

Just as before, we can classify topological spaces according to this relation, and obtain **classes of homotopy equivalence**:



the class of circles



the class of points

the class of spheres, the class of torii, the class of Klein bottles, ...

Example: Classification, up to homotopy equivalence, of the alphabet.

A B C D E F

G H I J K L

M N O P Q R

S T U V W X

Y Z

Classes de homotopia

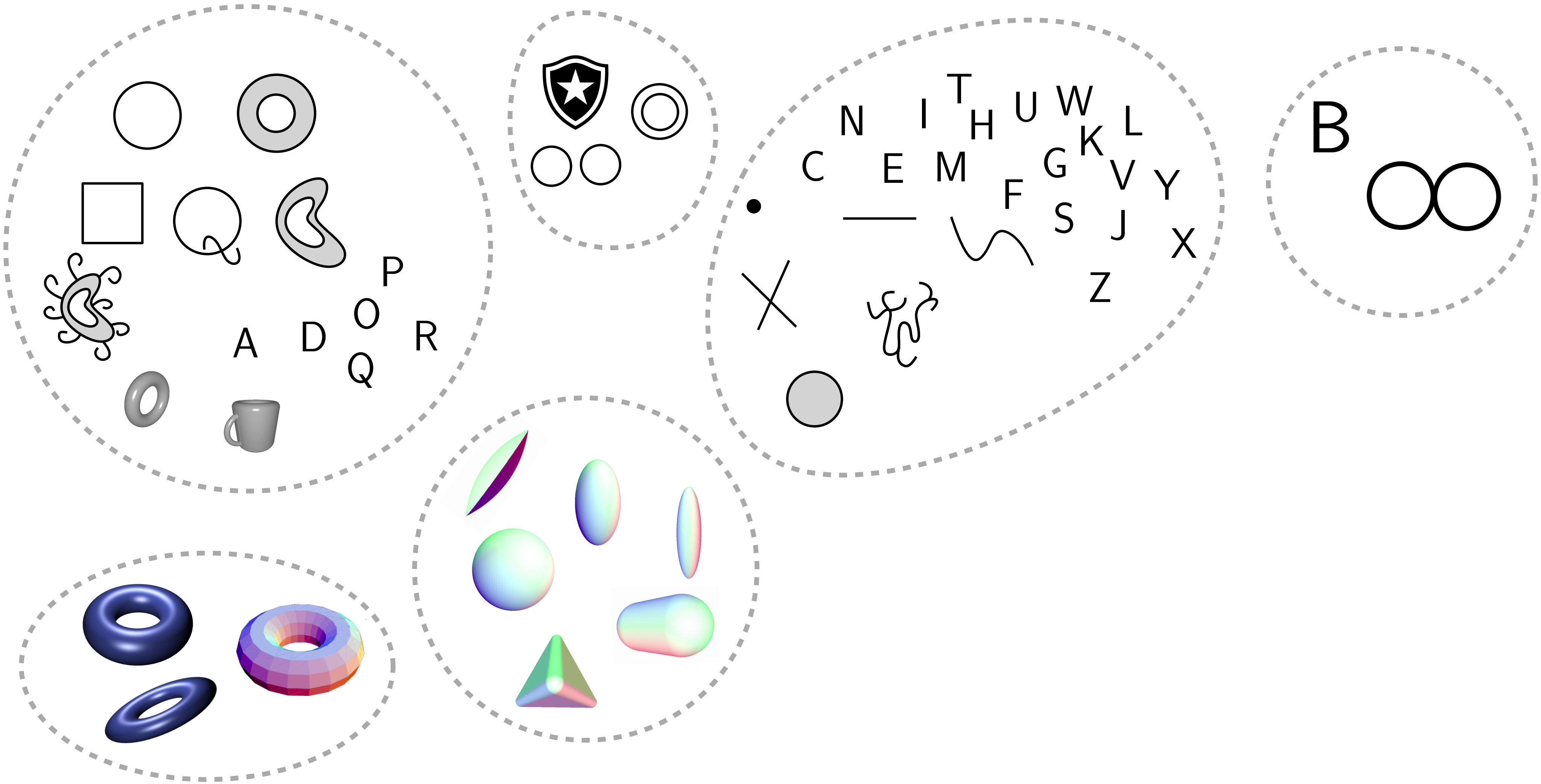
Example: Classification, up to homotopy equivalence, of the alphabet.

A D O P Q R $\approx \bigcirc$

B $\approx \bigcirc\!\!\bigcirc$

C E F G H I J K L
M N S T U V W X Y Z $\approx \bullet$

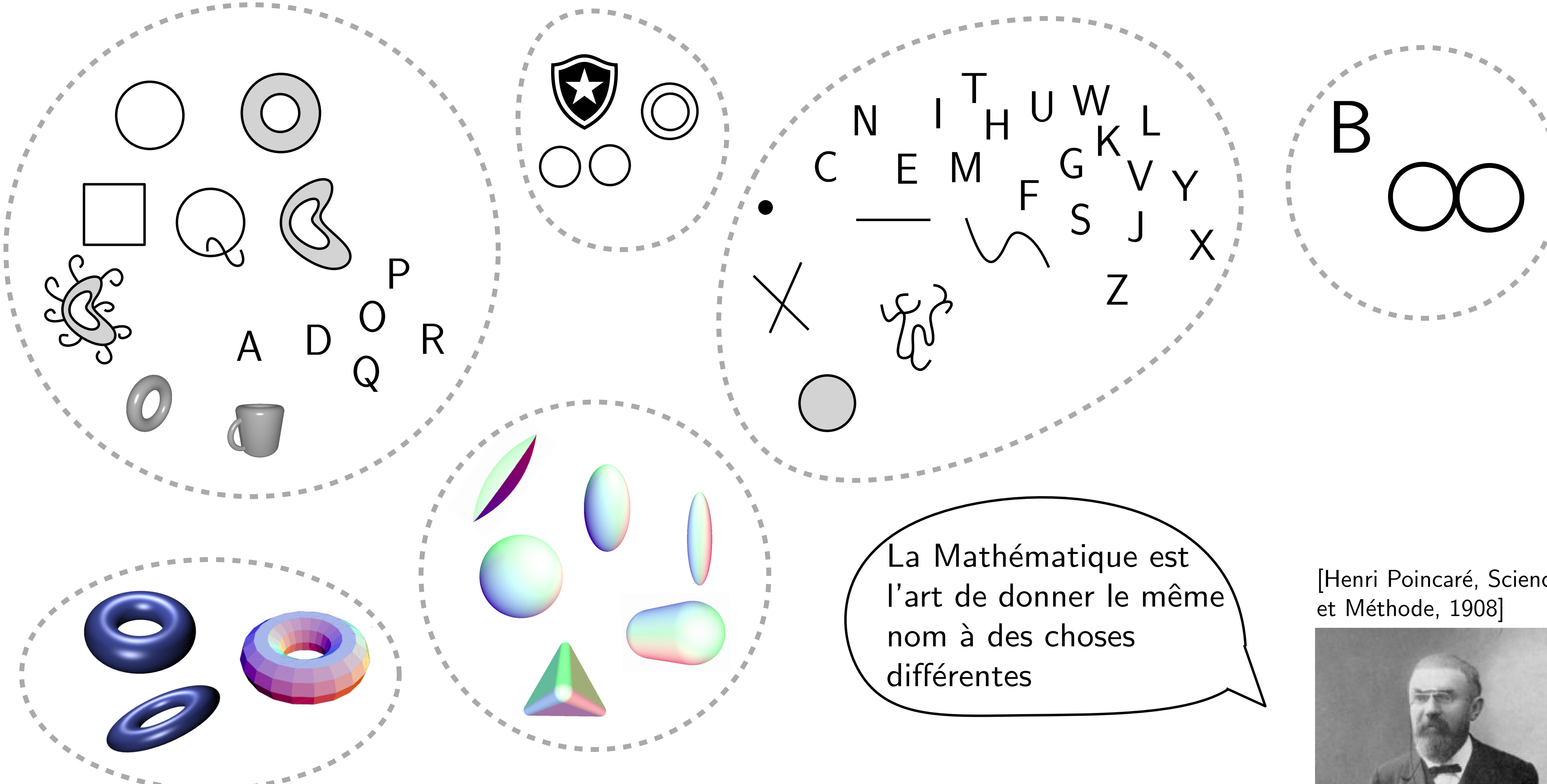
We gathered topological spaces into homotopy classes.



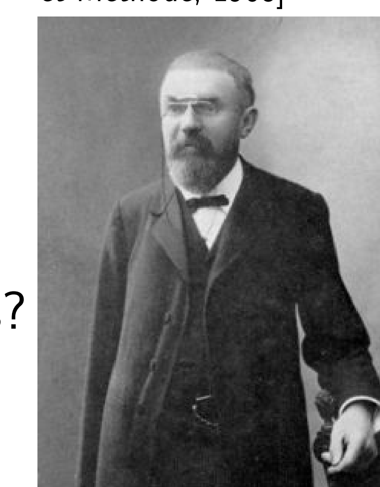
- Given a topological space X, how to recognize in which class it belongs?
- What are the common **features** of spaces in a same class?

Características comuns

We gathered topological spaces into homotopy classes.



[Henri Poincaré, Science et Méthode, 1908]



- Given a topological space X, how to recognize in which class it belongs?
- What are the common **features** of spaces in a same class?

I - Topology

- 1 - History
- 2 - Topological spaces

II - Comparing topological spaces

- 1 - Homeomorphism equivalence
- 2 - Homotopy equivalence

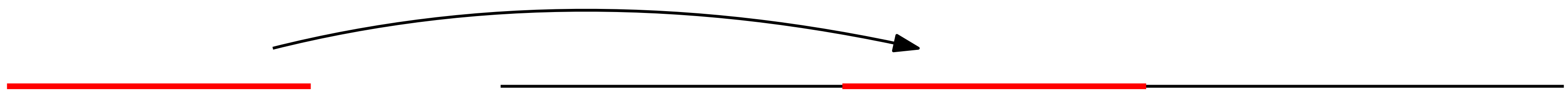
III - Topological invariants

- 1 - Embeddability
- 2 - Number of connected components
- 3 - Euler characteristic
- 4 - Betti numbers

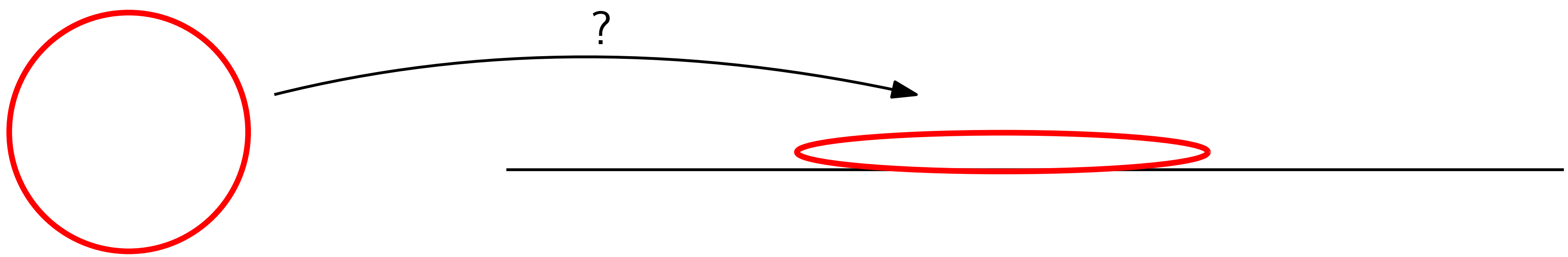
Imersibilidade

Definition: Let $n \in \mathbb{N}$. A topological space X is **embeddable** in $\mathbb{R}^n$ if there exists a continuous injective map $X \to \mathbb{R}^n$.

Example: The interval $(0,1)$ is embeddable in $\mathbb{R}$.



The circle $\mathbb{S}^1$ is not.



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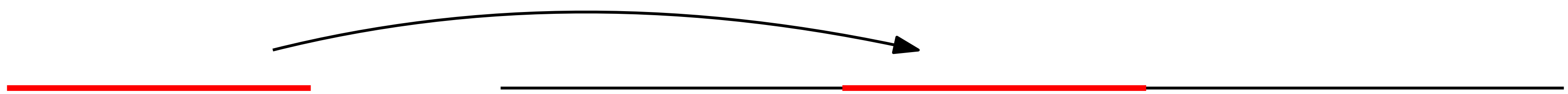

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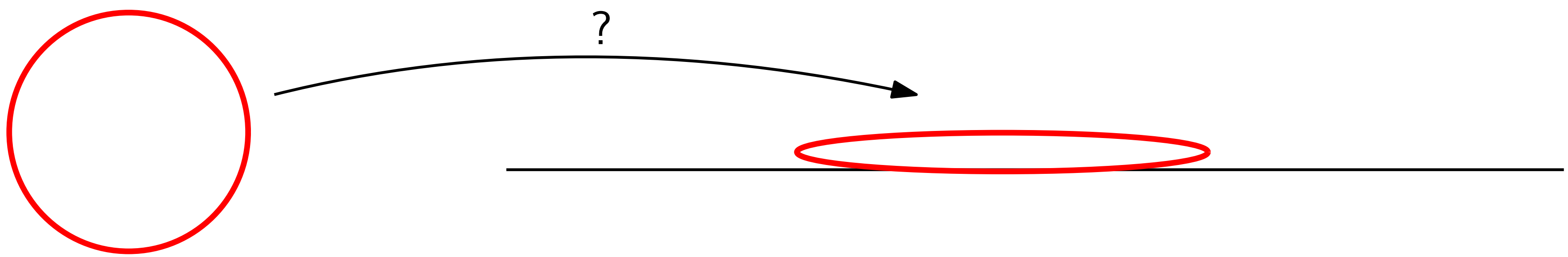
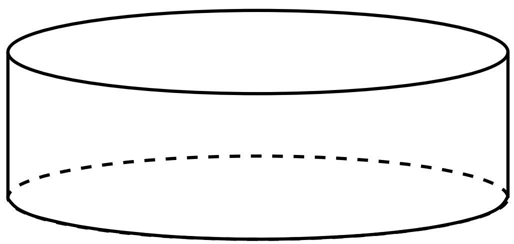


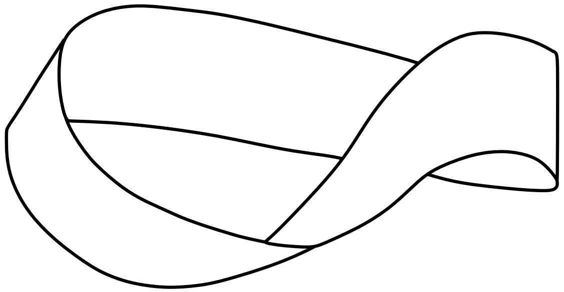
Proposition: Let $n \in \mathbb{N}$. If two spaces X and Y are homeomorphic, then either both are embeddable in $\mathbb{R}^n$, or neither.

We say that 'being embeddable in $\mathbb{R}^n$' is an **invariant of homeomorphism classes**. It can be used to show that two spaces are not homeomorphic.

Proposition: Let $n \in \mathbb{N}$. If two spaces X and Y are homeomorphic, then either both are embeddable in $\mathbb{R}^n$, or neither.

Example: The cylinder and the Möbius strip are not homeomorphic.





Indeed, the cylinder can be embedded in $\mathbb{R}^2$ (as an annulus).

If the strip was homeomorphic to the cylinder, then it would be also embeddable in $\mathbb{R}^2$.

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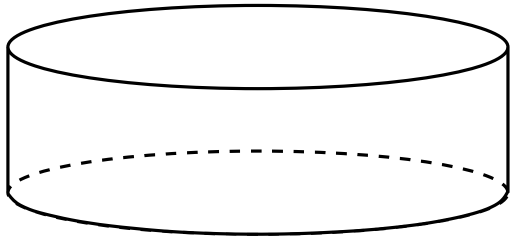

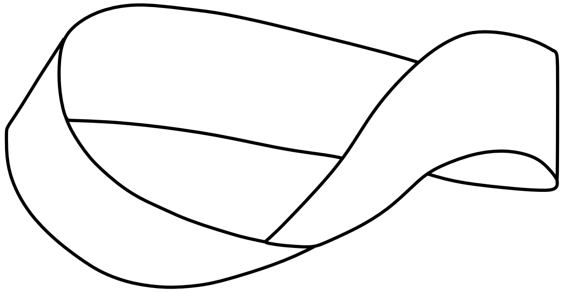

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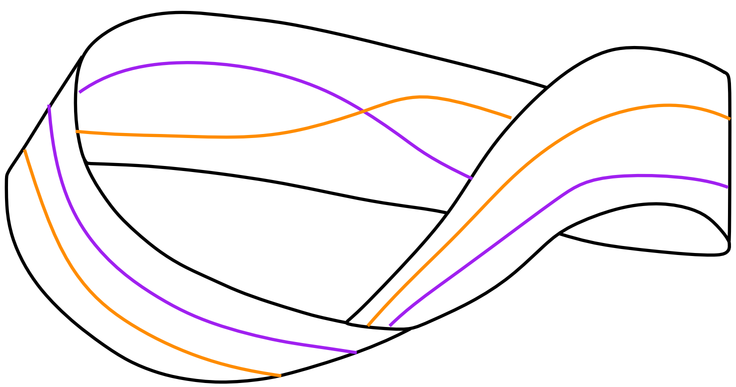
We draw two circles on the strip, C_1 and C_2, that only intersect once.



Propriedade de invariância - na teoria

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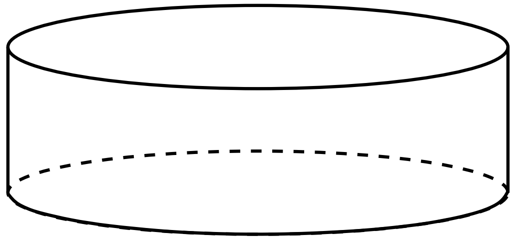

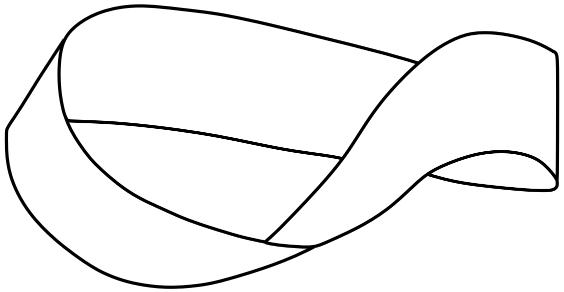

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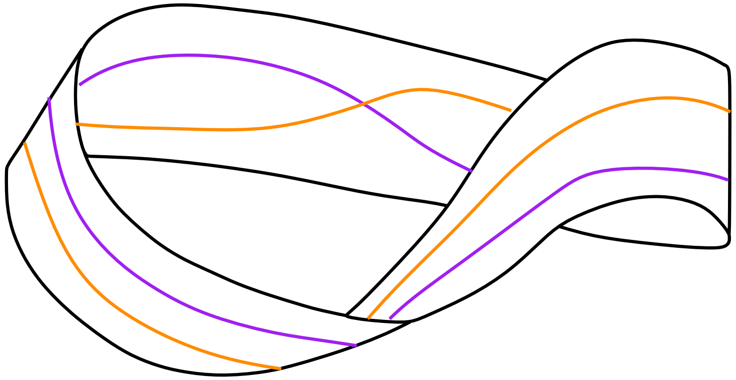

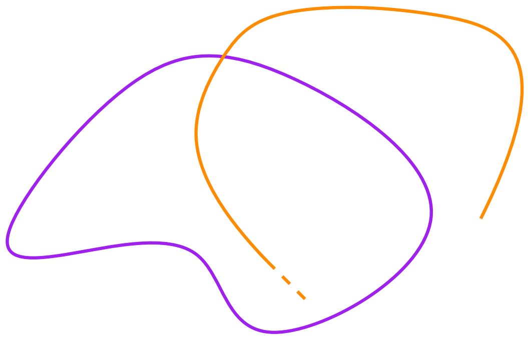

Embedded in $\mathbb{R}^2$, the circles C^1 and C^2 only intersect once.
This is impossible by Jordan's theorem.

Proposition: Let $n \in \mathbb{N}$. If two spaces X and Y are homeomorphic, then either both are embeddable in $\mathbb{R}^n$, or neither.

Example: The cylinder and the Möbius strip are not homeomorphic.

Remark: The property 'being embeddable in $\mathbb{R}^n$' is **not** an invariant of *homotopy* classes.

Indeed, the space and the cylinder are homotopy equivalent, but only one of them is embeddable in $\mathbb{R}^2$.

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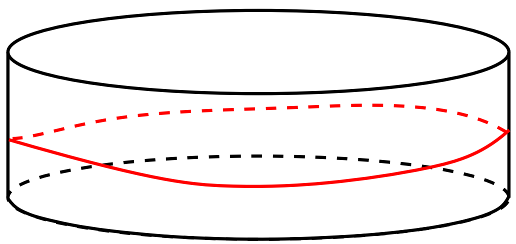

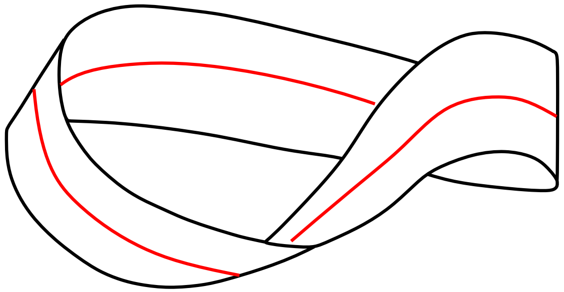

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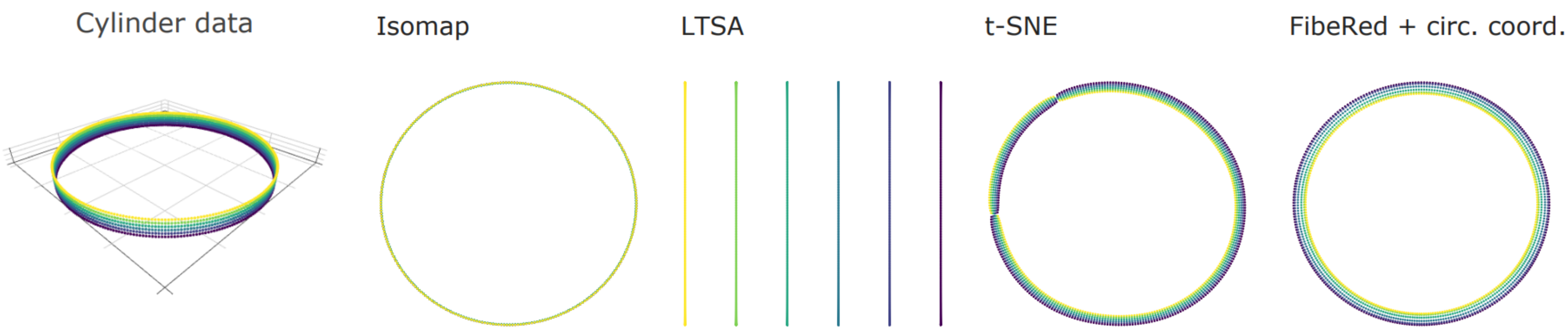
Indeed, the space and the cylinder are homotopy equivalent, but only one of them is embeddable in $\mathbb{R}^2$.

They can both be retracted onto their inner circle.

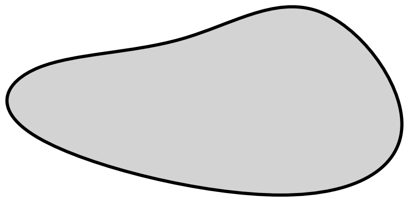
In applications, finding an embedding corresponds to the problem of **dimensionality reduction**.



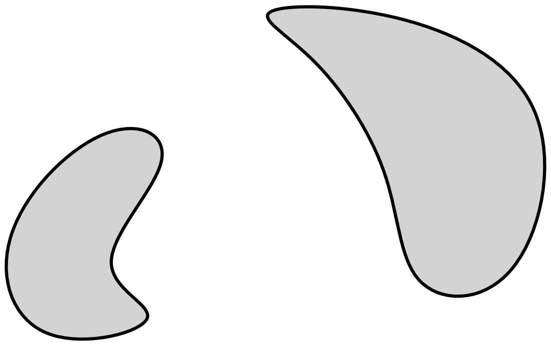
Illustrations from [Luis Scoccola, Jose A. Perea, Fiberwise dimensionality reduction of topologically complex data with vector bundles, 2022]

Componentes conexas

Definition: A subset $X \subset \mathbb{R}^n$ is (path-) **connected** if for every $x, y \in X$, there exists a continuous map $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.



connected space



non-connected space

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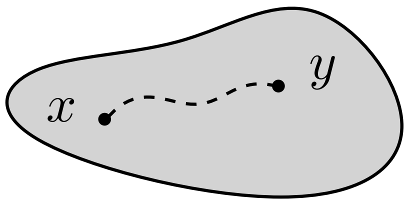
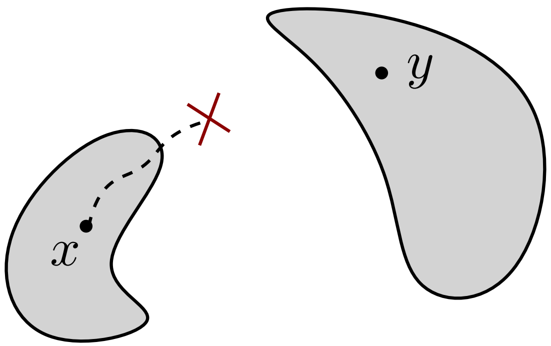


connected space



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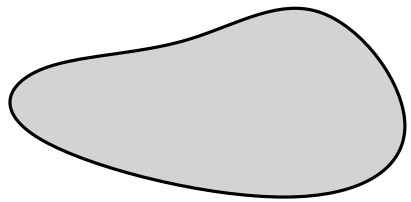
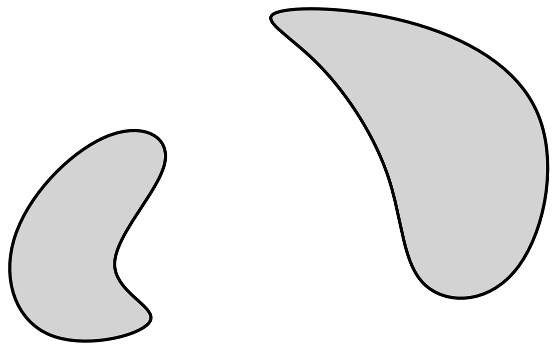
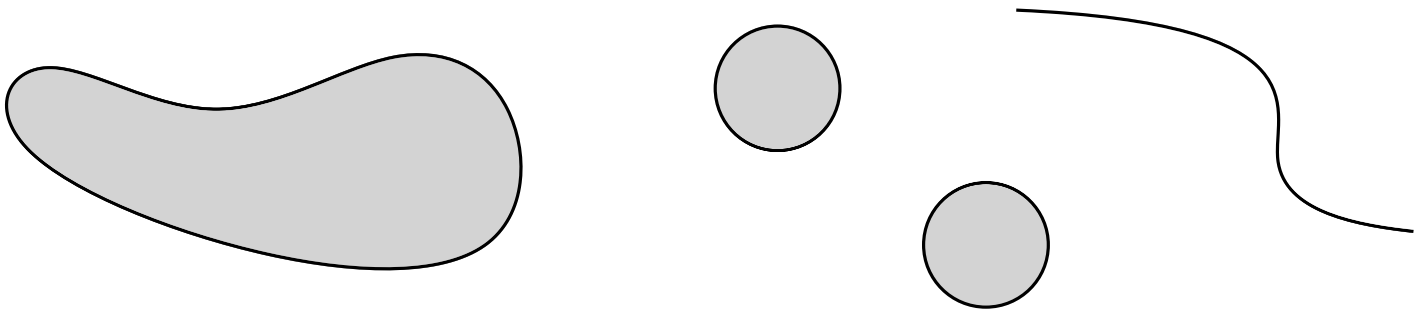


connected space



non-connected space

More generally, any topological space X can be partitioned into **connected components**.



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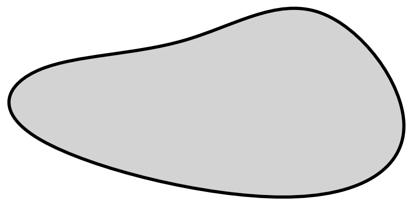
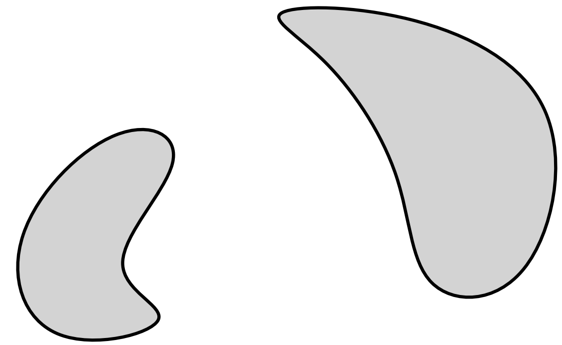


connected space



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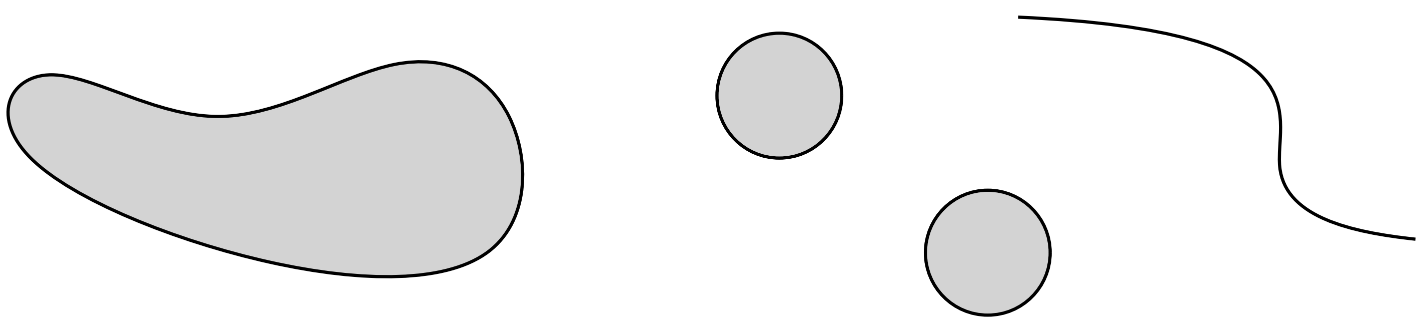


Proposition: If two spaces X and Y are homotopy equivalent, then they have the same number of connected components.

Propriedade de invariância - na teoria

Proposition: If two spaces X and Y are homotopy equivalent, then they have the same number of connected components.

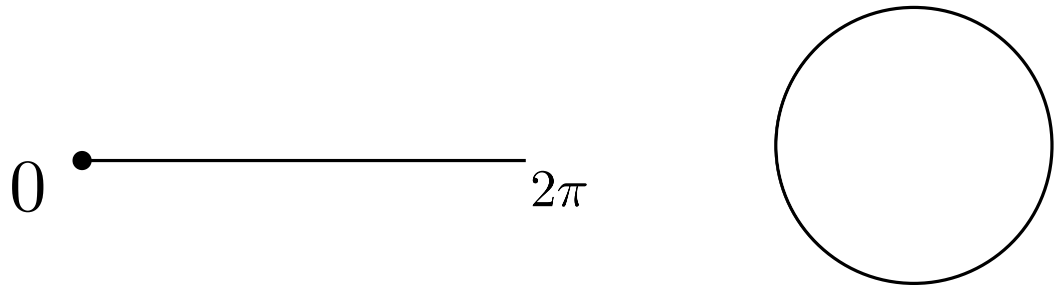
Consequence: If two spaces X and Y are homeomorphic, then they have the same number of connected components.

Example: The subsets $[0, 1]$ and $[0, 1] \cup [2, 3]$ of $\mathbb{R}$ are not homeomorphic, neither homotopy equivalent.
Indeed, the first one has one connected component, and the second one two.

Proposition: If two spaces X and Y are homotopy equivalent, then they have the same number of connected components.

Example: The interval $[0, 2\pi)$ and the circle $\mathbb{S}_1 \subset \mathbb{R}^2$ are not homeomorphic.

We will prove this by contradiction. Suppose that they are homeomorphic. By definition, this means that there exists a map $f\colon [0, 2\pi) \to \mathbb{S}_1$ which is continuous, invertible, and with continuous inverse.



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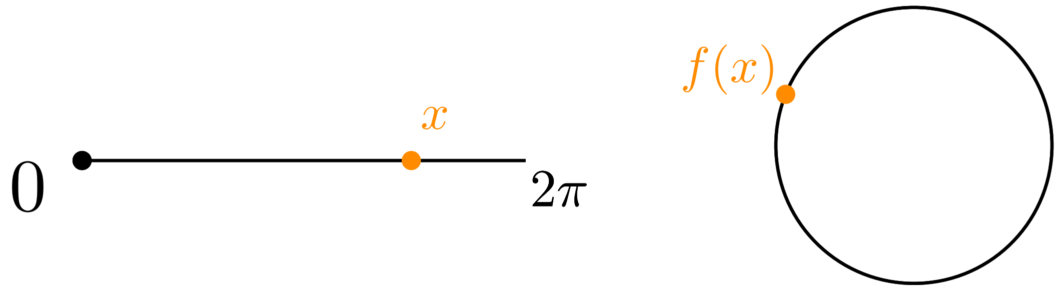


Let $x \in [0, 2\pi)$ such that $x \neq 0$. Consider the subsets $[0, 2\pi) \setminus \{x\} \subset [0, 2\pi)$ and $\mathbb{S}_1 \setminus \{f(x)\} \subset \mathbb{S}_1$, and the induced map

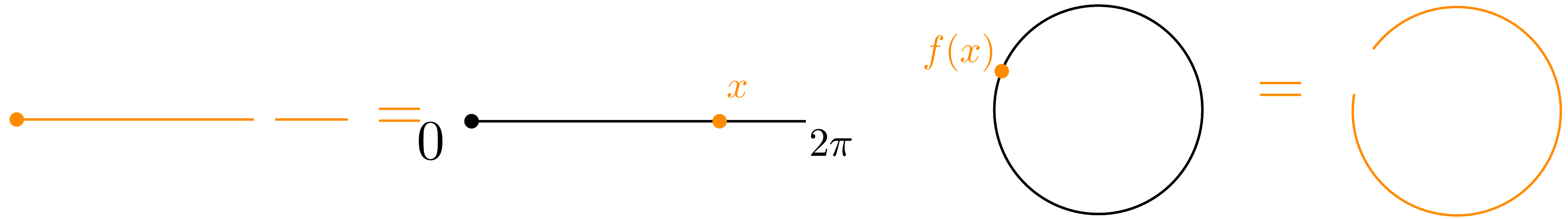
$$g\colon [0, 2\pi) \setminus \{x\} \to \mathbb{S}_1 \setminus \{f(x)\}.$$

The map g is a homeomorphism.

Proposition: If two spaces X and Y are homotopy equivalent, then they have the same number of connected components.

Example: The interval $[0, 2\pi)$ and the circle $\mathbb{S}_1 \subset \mathbb{R}^2$ are not homeomorphic.

We will prove this by contradiction. Suppose that they are homeomorphic. By definition, this means that there exists a map $f\colon [0, 2\pi) \to \mathbb{S}_1$ which is continuous, invertible, and with continuous inverse.



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$$g\colon [0, 2\pi) \setminus \{x\} \to \mathbb{S}_1 \setminus \{f(x)\}.$$

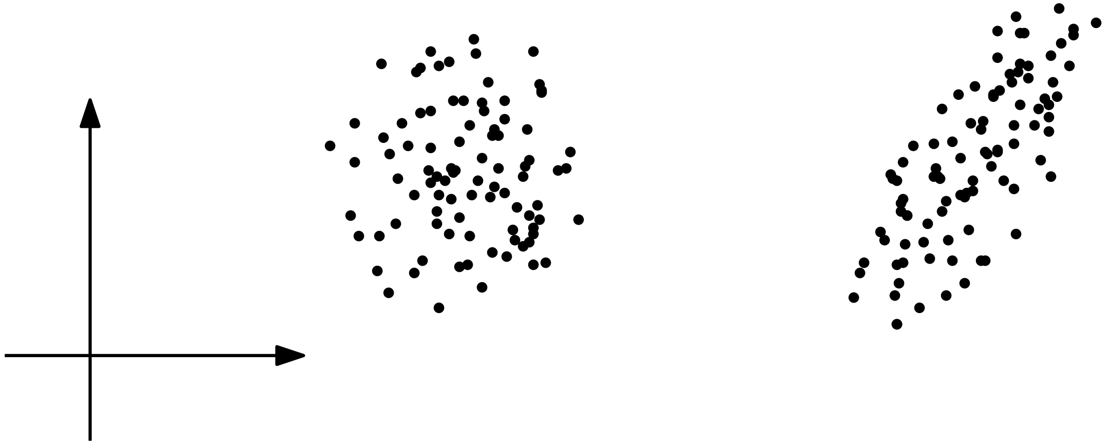
The map g is a homeomorphism.

Moreover, $[0, 2\pi) \setminus \{x\}$ has two connected components, and $\mathbb{S}_1 \setminus \{f(x)\}$ only one. This is absurd.

Proposition: If two spaces X and Y are homotopy equivalent, then they have the same number of connected components.

Homework: The intervals $[0, 1)$ and $(0, 1)$ are not homeomorphic.

In applications, finding connected components corresponds to a **classification** task.



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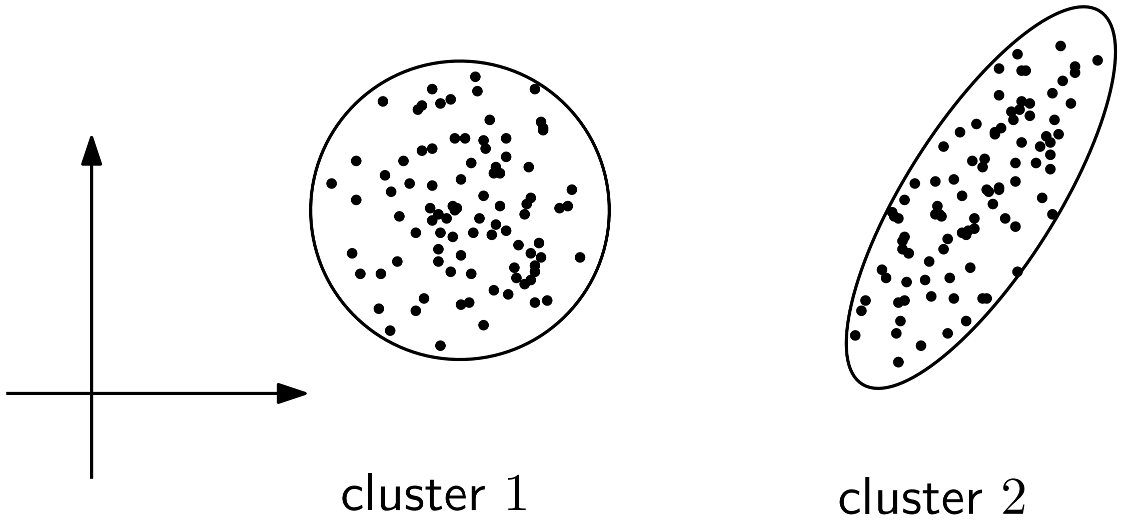

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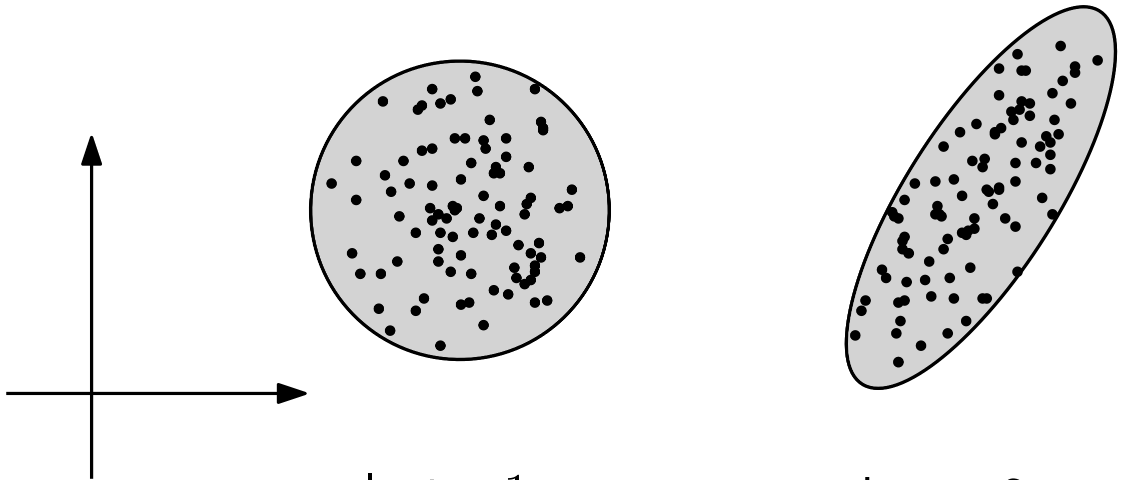
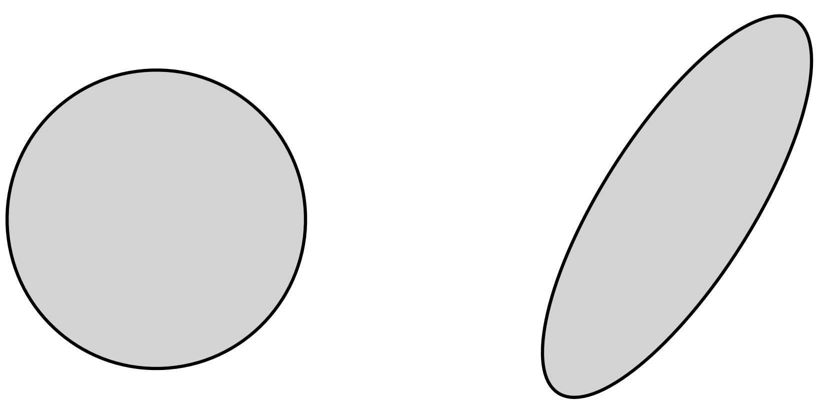


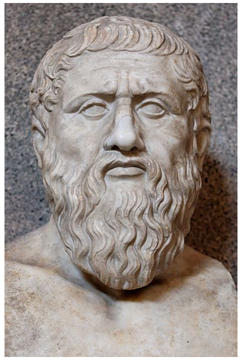
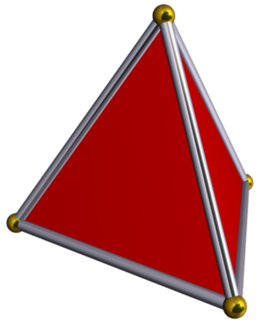
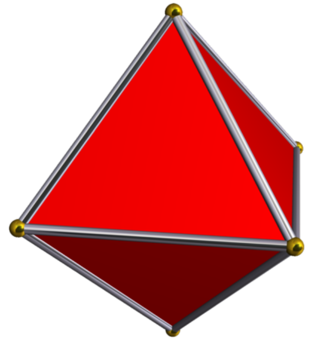
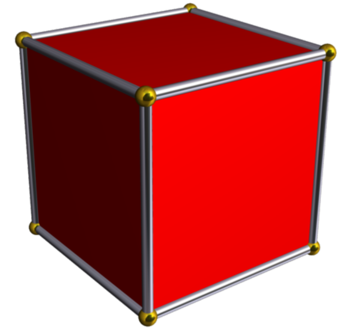
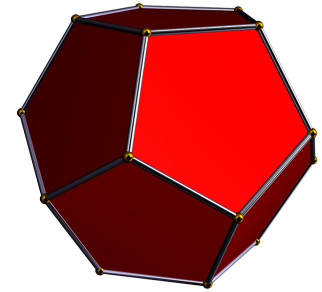
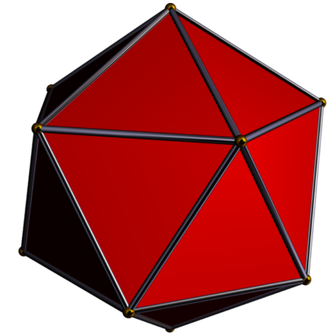
cluster 1 cluster 2



connected component 1 connected component 2

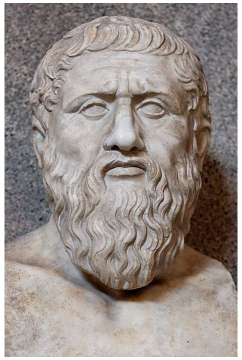
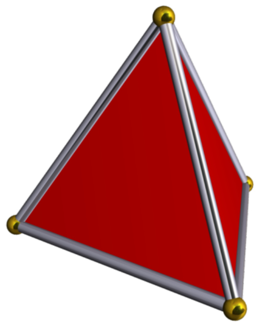
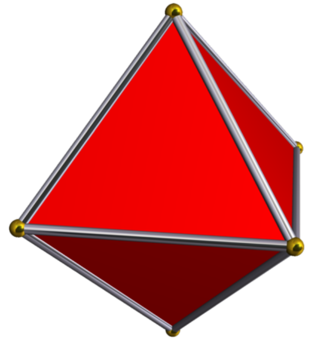
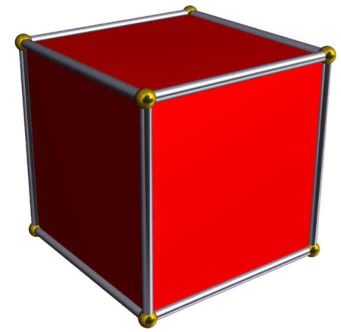
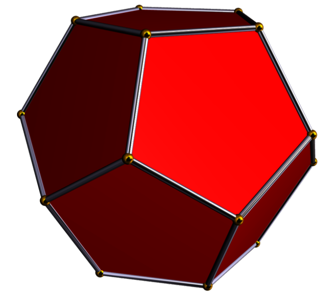
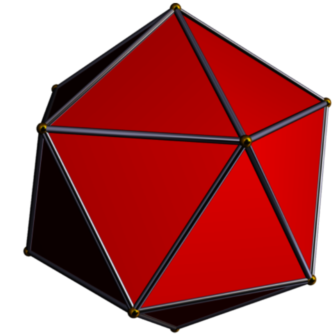
We can think of these sets as an **underlying topological space**, on which the points are sampled.

Característica de Euler

number of faces	4	8	6	12	20
number of edges	6	12	12	30	30
number of vertices	4	6	8	20	12
χ	2	2	2	2	2

Característica de Euler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Proposition [Euler, 1758]: In any convex polyhedron, we have
number of faces $-$ number of edges $+$ number of vertices $= 2$

Definition: Let V be a set (called the set of *vertices*). A **simplicial complex** over V is a set K of subsets of V (called the *simplices*) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$.

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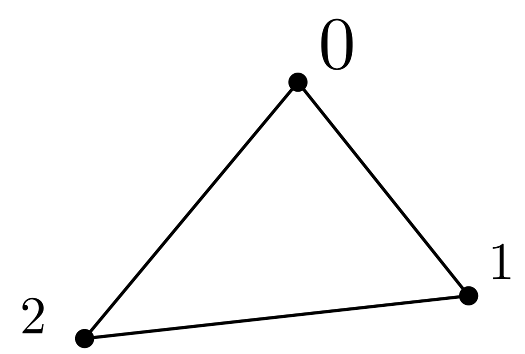
Example: Let $V = \{0, 1, 2\}$ and

$$K = \{[0], [1], [2], [0, 1], [1, 2], [0, 2]\}.$$

This is a simplicial complex.



It contains three simplices of dimension 0 ($[0]$, $[1]$ and $[2]$) and three simplices of dimension 1 ($[0, 1]$, $[1, 2]$ and $[0, 2]$).

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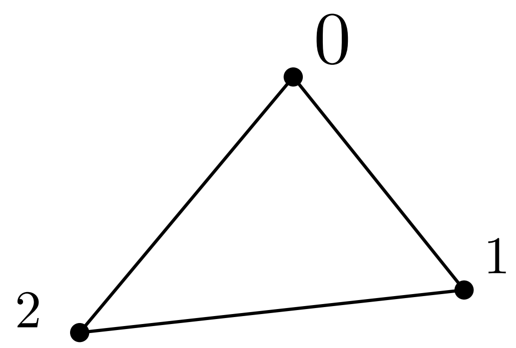

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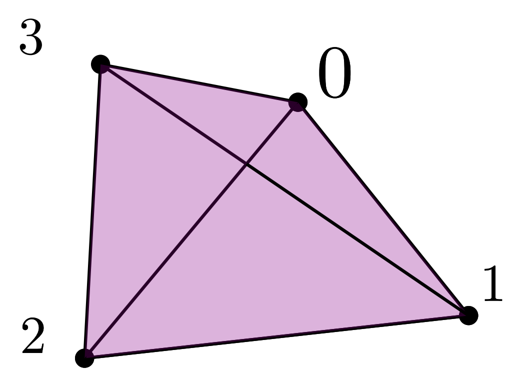
Example: Let $V = \{0, 1, 2, 3\}$ and

$$K = \{[0], [1], [2], [3], [0,1], [1,2], [2,3], [3,0], [0,2], [1,3], [0,1,2], [0,1,3], [0,2,3], [1,2,3]\}$$

It a simplicial complex.



It contains four simplices of dimension 0 ($[0]$, $[1]$, $[2]$ and $[3]$), six simplices of dimension 1 ($[0,1]$, $[1,2]$, $[2,3]$, $[3,0]$,$[0,2]$ and $[1,3]$) and four simplices of dimension 2 ($[0,1,2]$, $[0,1,3]$, $[0,2,3]$ and $[1,2,3]$).

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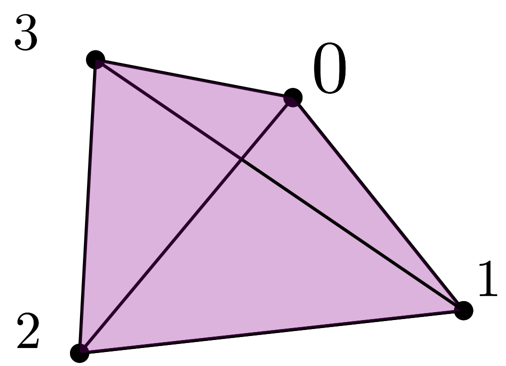
It a simplicial complex.



(this is a sphere)

It contains four simplices of dimension 0 ($[0]$, $[1]$, $[2]$ and $[3]$), six simplices of dimension 1 ($[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 0]$,$[0, 2]$ and $[1, 3]$) and four simplices of dimension 2 ($[0, 1, 2]$, $[0, 1, 3]$, $[0, 2, 3]$ and $[1, 2, 3]$).

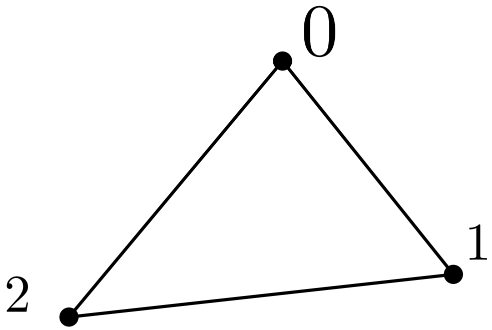
Característica de Euler

Definition: Let K be a simplicial complex of dimension n. Its **Euler characteristic** is the integer

$$\chi(K) = \sum_{0 \le i \le n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Example: The simplicial complex $K = \{[0], [1], [2], [0,1], [1,2], [2,0]\}$ has Euler characteristic

$$\chi(K) = 3 - 3 = 0$$



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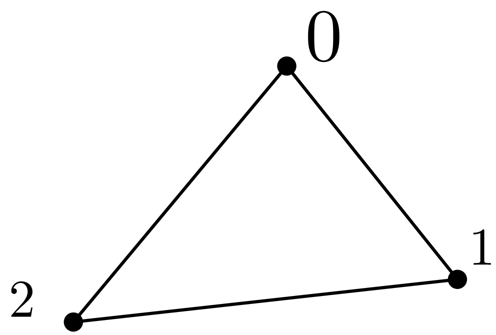
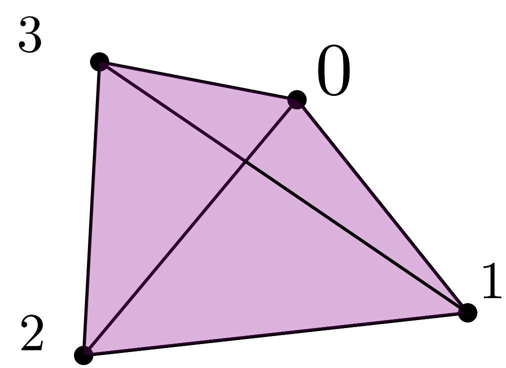


Example: The simplicial complex
$K = \{[0],[1],[2],[3],[0,1],[1,2],[2,3],[3,0],[0,2],[1,3],[0,1,2],[0,1,3],[0,2,3],[1,2,3]\}$
has Euler characteristic

$$\chi(K) = 4 - 6 + 4 = 2$$



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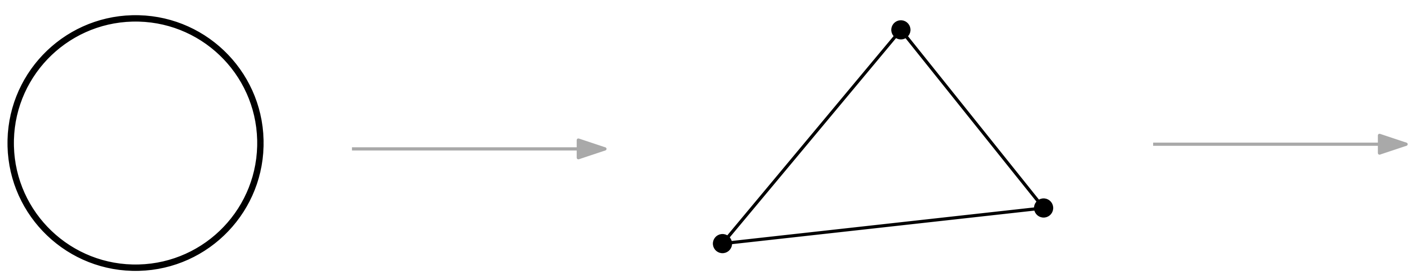
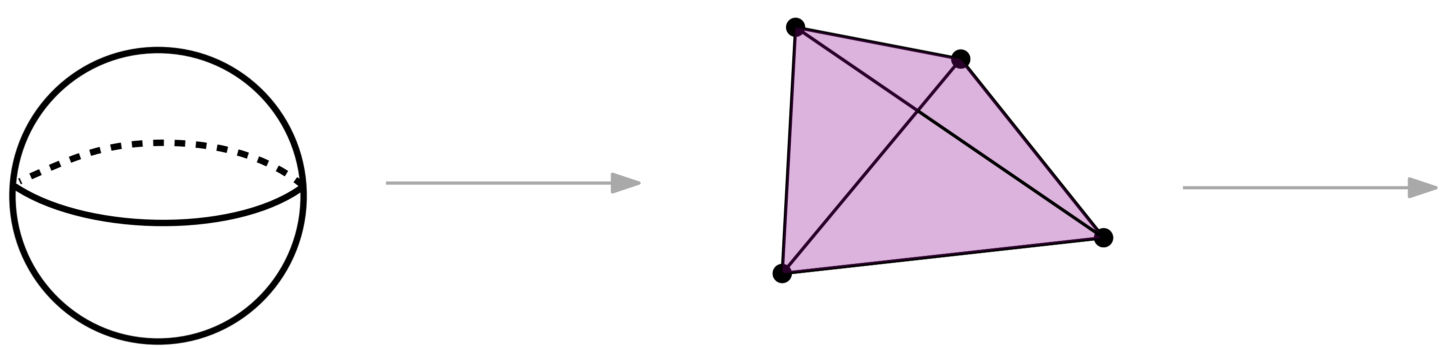
Definition: Let X be a topological space. Its **Euler characteristic** is defined as the Euler caracteristic of a *triangulation* of X.



$\chi(X) = 0$



$\chi(X) = 2$

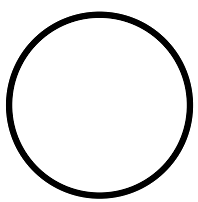
Propriedade de invariância - na teoria

Proposition: If X and Y are two homotopy equivalent topological spaces, then $\chi(X) = \chi(Y)$.

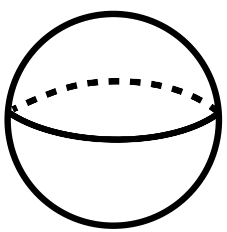
Therefore, the Euler characteristic is an **invariant** of homotopy equivalence classes.

We can use this information to prove that two spaces are not homotopy equivalent.

Example: The circle has Euler characteristic 0, and the sphere Euler characteristic 2. Therefore, they are not homotopy equivalent.



$\chi(\mathbb{S}_1) = 0$



$\chi(\mathbb{S}_2) = 2$

Propriedade de invariância - na teoria

Proposition: If X and Y are two homotopy equivalent topological spaces, then $\chi(X) = \chi(Y)$.

Therefore, the Euler characteristic is an **invariant** of homotopy equivalence classes.

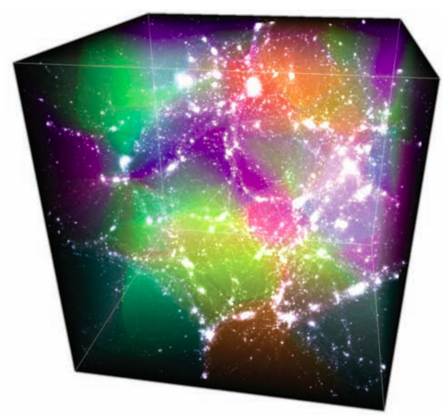
We can use this information to prove that two spaces are not homotopy equivalent.

Example (Classification of surfaces): The homeomorphism classes of *connected and compact surfaces* are classified by their Euler characteristic.

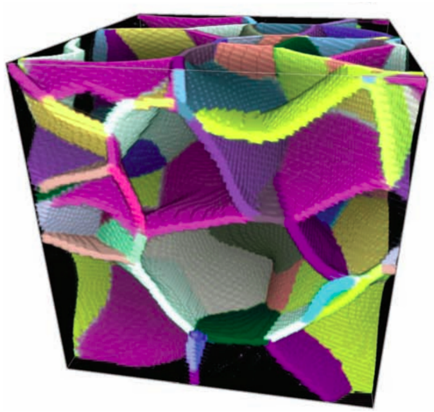
$\chi \quad 2 \quad 0 \quad -2 \quad -4 \quad 2 - 2 \times \text{genus}$

The Euler characteristic contains information about the homeomorphism class (and homotopy class) of the space.

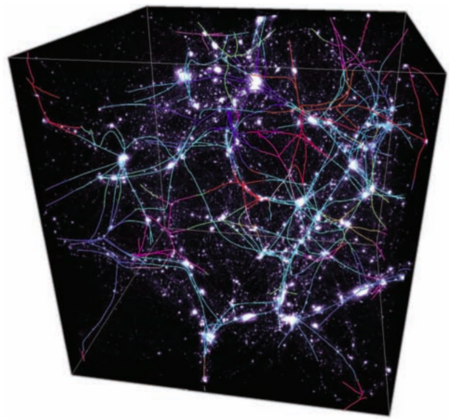
[T. Sousbie, The persistent cosmic web and its filamentary structure, 2011]



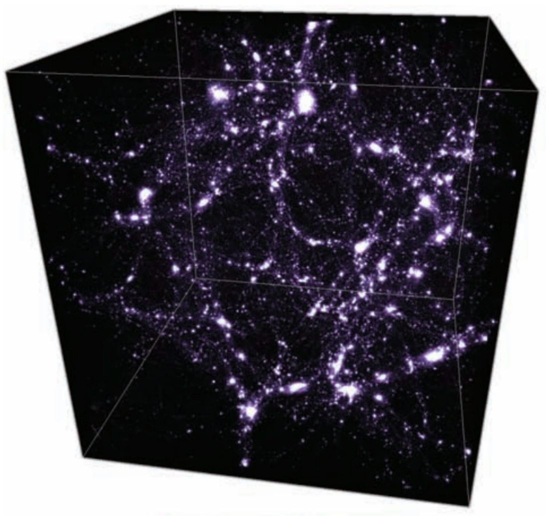
seen as an object of dimension 3



of dimension 2



of dimension 1



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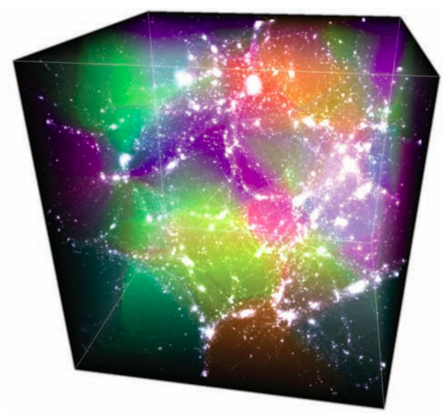
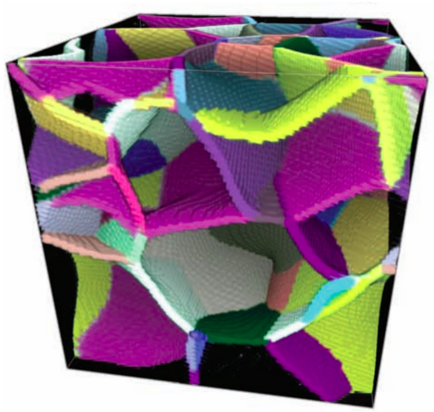
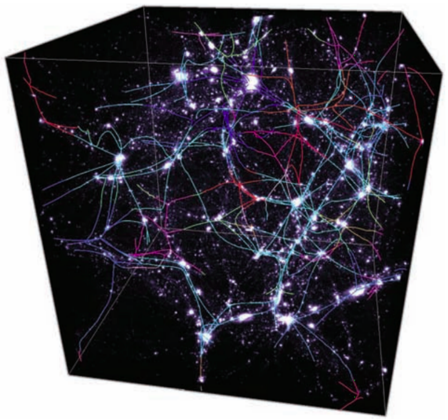
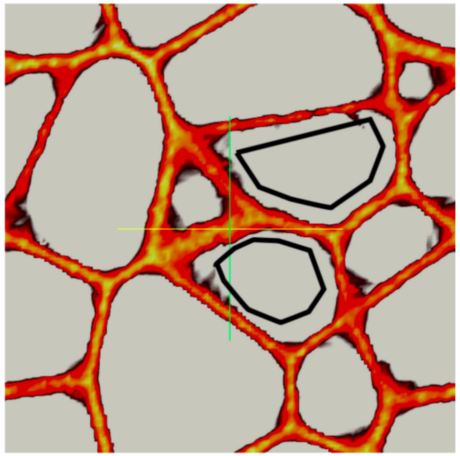






[P. Pranav, H. Edelsbrunner, R. de Weygaert, G. Vegter, M. Kerber, B. Jones and M. Wintraecken, The topology of the cosmic web in terms of persistent Betti numbers, 2016]



The Euler characteristic 'counts' the number of holes

Números de Betti

For any topological space X, one defines a sequence of integers

$$\beta_0(X), \quad \beta_1(X), \quad \beta_2(X), \quad \beta_3(X), \quad \ldots$$

called the **Betti numbers**.

Construction of Betti numbers:

Números de Betti

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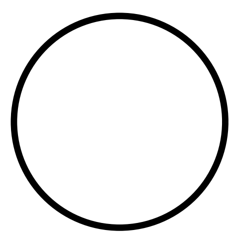
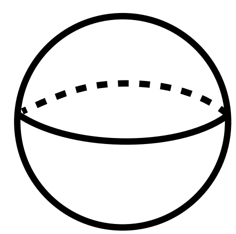
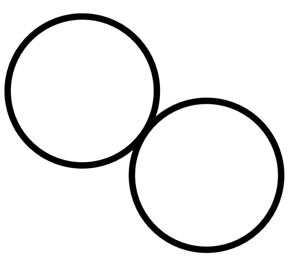
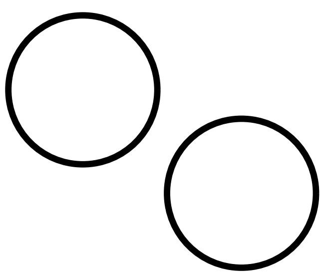
Construction of Betti numbers: rendez-vous on Thursday! (based on homology theory)

Example: Let us give some examples instead.

X					
$\beta_0(X)$	1	1	1	1	2
$\beta_1(X)$	0	1	0	2	2
$\beta_2(X)$	0	0	1	0	0

Números de Betti

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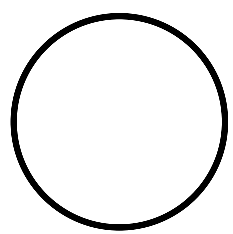
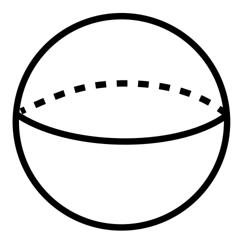
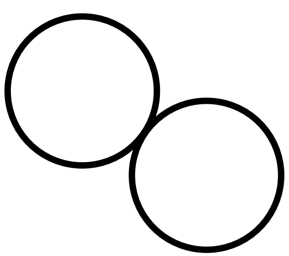
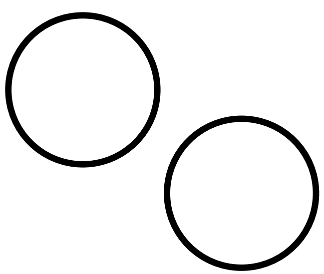
X					
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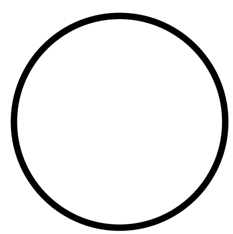
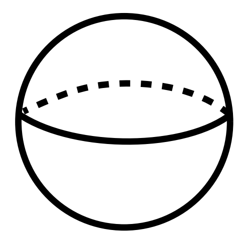
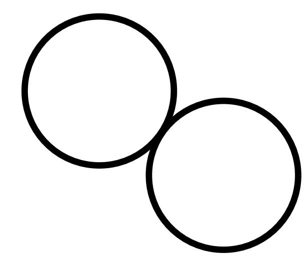
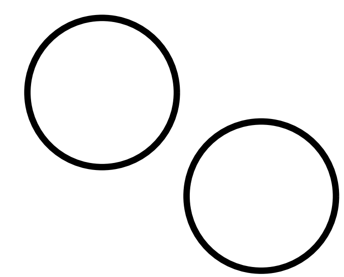
Números de Betti

Interpretation: We have:

- $\beta_0(X)$ is the number of connected components of X
- $\beta_1(X)$ is the number of 'holes' in X
- $\beta_2(X)$ is the number of 'voids' in X
- $\dots$

Example: Let us give some examples instead.

X					
$\beta_0(X)$	1	1	1	1	2
$\beta_1(X)$	0	1	0	2	2
$\beta_2(X)$	0	0	1	0	0

Números de Betti

Interpretation: We have:

- $\boxed{\beta_0(X) \text{ is the number of connected components of } X}$
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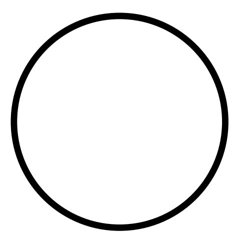
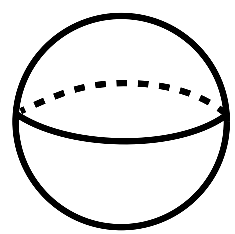
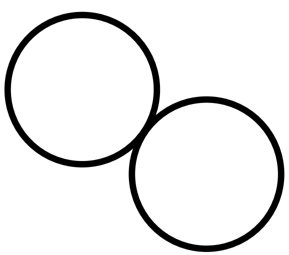
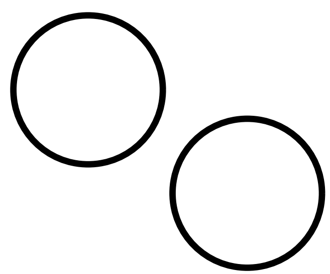
X					
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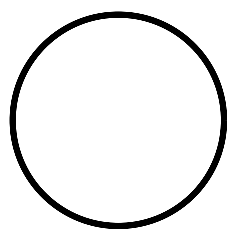
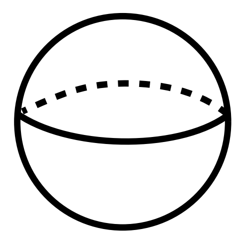
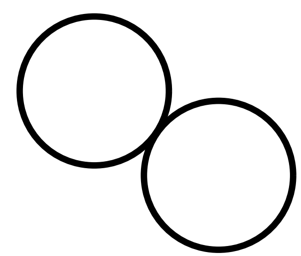
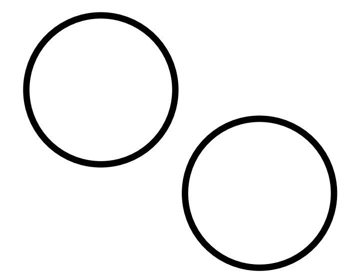
X					
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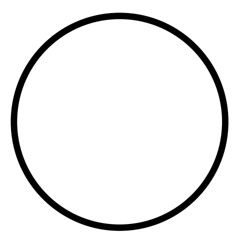
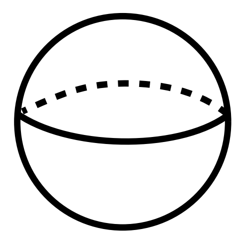
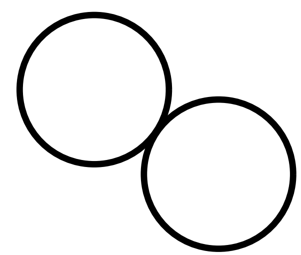
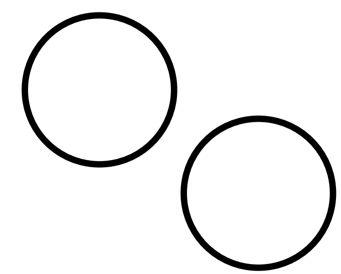
X					
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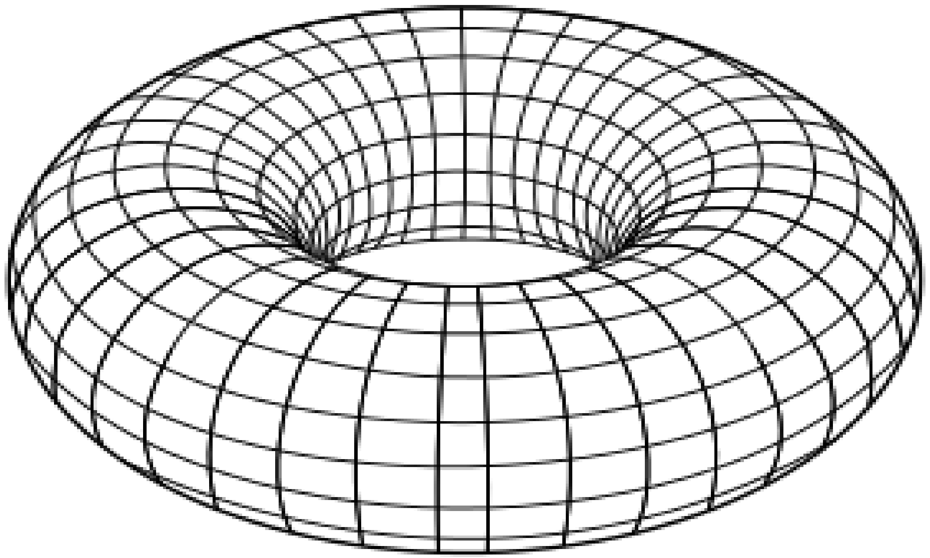
Example: Betti numbers of the torus:

$$\beta_0(X) = 1, \quad \beta_1(X) = 2, \quad \beta_2(X) = 1, \quad \beta_3(X) = 0, \quad \ldots$$



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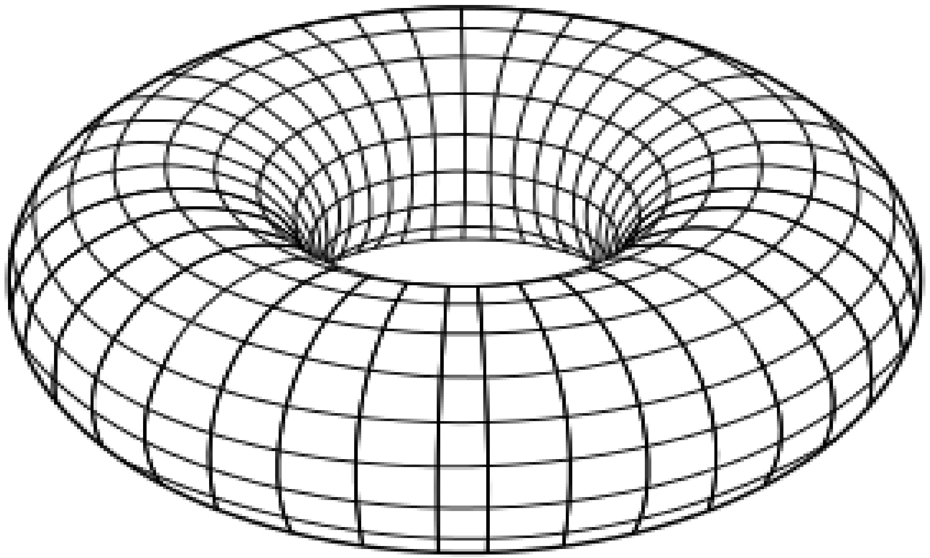

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Example: Betti numbers of the torus:

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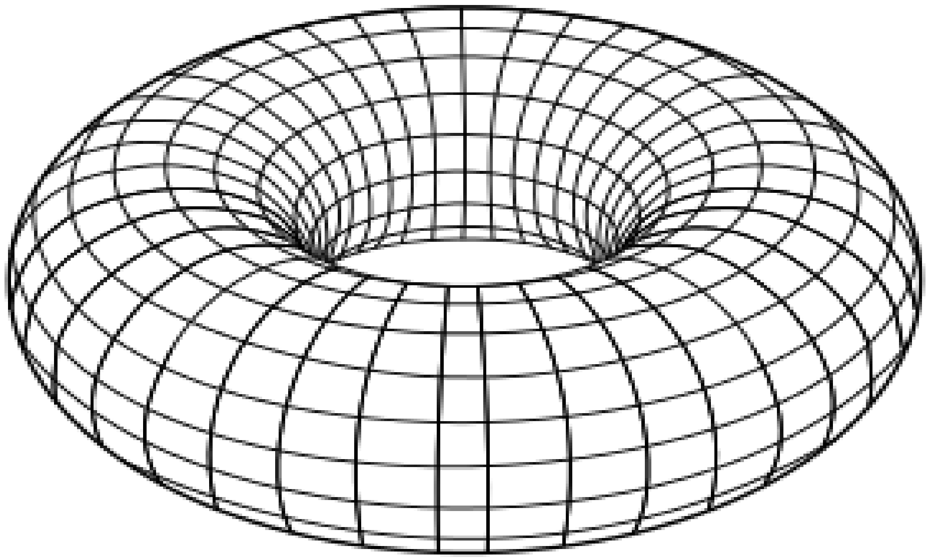

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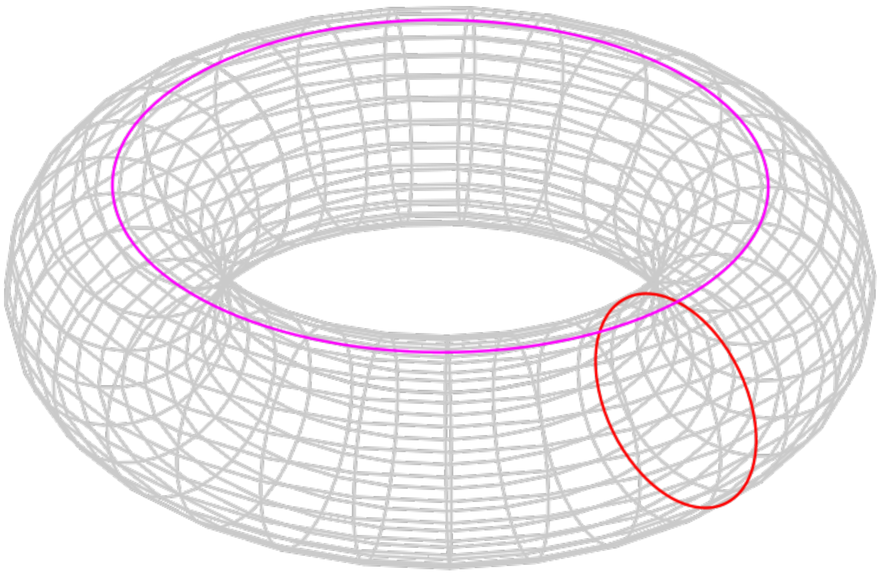


Proposition: If two spaces X and Y are homotopy equivalent, then they have the same Betti numbers.

As a consequence, two spaces with different Betti numbers cannot be homotopy equivalent.

Example: The n-dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has Betti numbers

$$\begin{aligned} \beta_i(X) &= 1 \quad \text{if } \ i = 0 \text{ or } n, \\ \beta_i(X) &= 0 \quad \text{else.} \end{aligned}$$

Hence, if $n \neq m$, then $\mathbb{S}^n$ and $\mathbb{S}^n$ are not homotopy equivalent.

Proposition: If two spaces X and Y are homotopy equivalent, then they have the same Betti numbers.

As a consequence, two spaces with different Betti numbers cannot be homotopy equivalent.

Example (Brouwer's invariance of domain):

Let us show that $\mathbb{R}^n$ and $\mathbb{R}^m$, with $n \neq m$, are not homeomorphic.

Let $h\colon \mathbb{R}^n \to \mathbb{R}^m$ be a homeomorphism.
Choose any $x \in \mathbb{R}^n$and consider the restricted map

$$h\colon \mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R}^m \setminus \{h(x)\}$$

It is still a homemorphism.

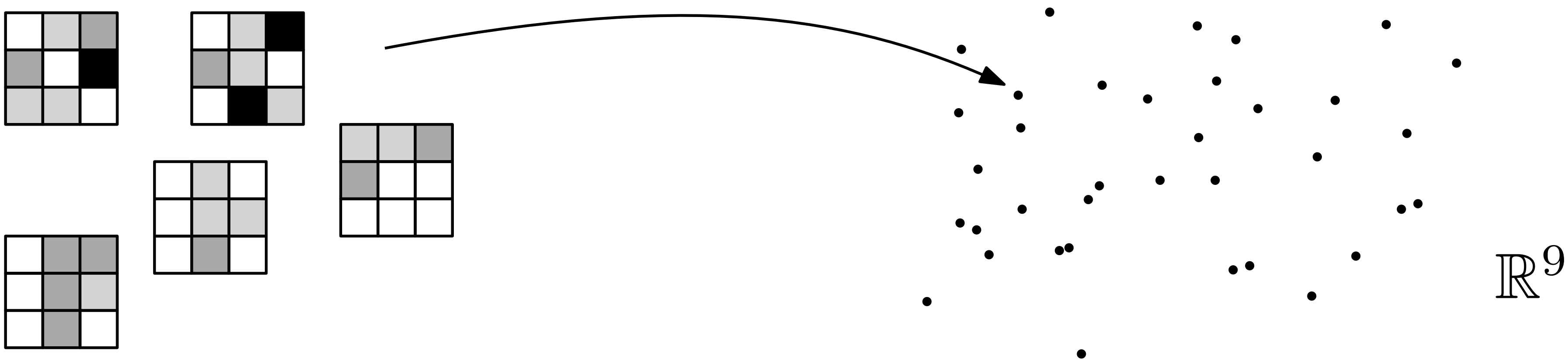
But $\mathbb{R}^n \setminus \{x\}$ is homotopic to the sphere $\mathbb{S}^{n-1}$, and $\mathbb{R}^m \setminus \{x\}$ is homotopic to the sphere $\mathbb{S}^{m-1}$

We have seen before that $\mathbb{S}^{n-1}$ and $\mathbb{S}^{m-1}$ are homotopic if and only if $m = n$. This is a contradiction.

The Betti numbers contain information about the space we study.

[G. Carlsson, T. Ishkhanov, V. de Silva, and A. Zomorodian, On the Local Behavior of Spaces of Natural Images, 2008.]

From a large collection of natural images, the authors extract 3×3 patches.
Since it consists of 9 pixels, each of these patches can be seen as a 9-dimensional vector, and the whole set as a point cloud in $\mathbb{R}^9$.



They observe that the point cloud lies close to a shape whose Betti numbers (over $\mathbb{Z}/2\mathbb{Z}$) are

$$\beta_0(X) = 1, \quad \beta_1(X) = 2, \quad \beta_2(X) = 1, \quad \beta_3(X) = 0$$

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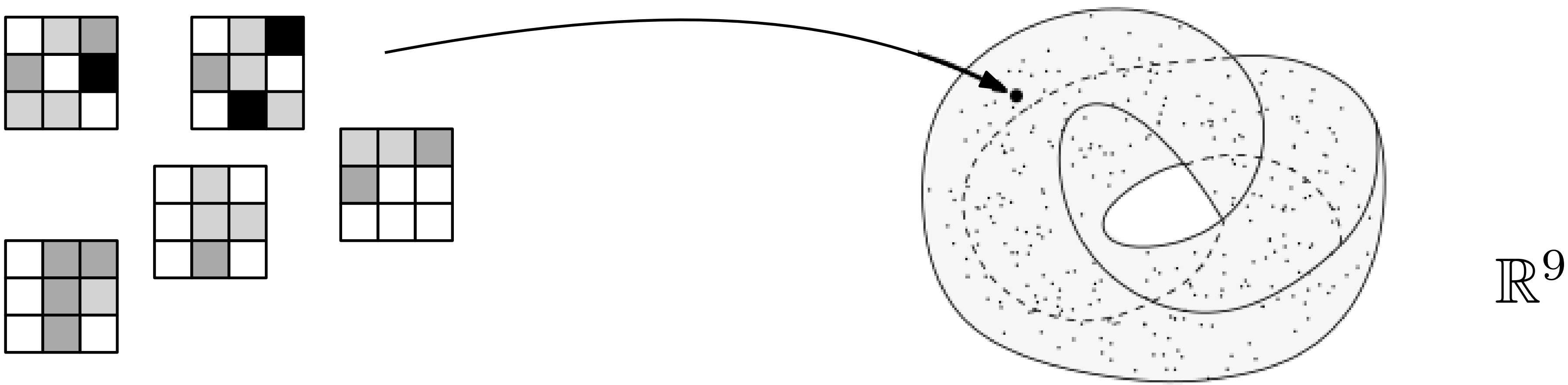

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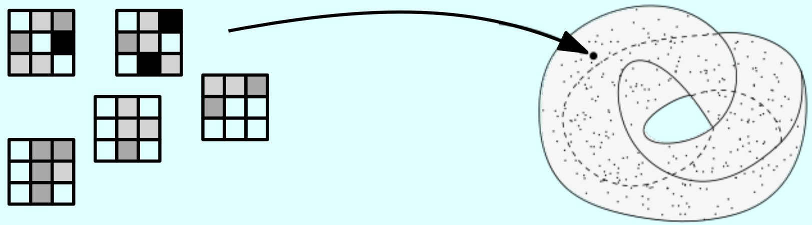
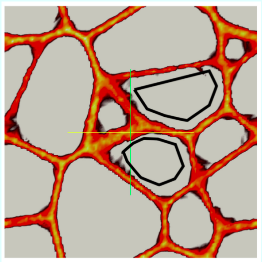
These are the Betti numbers of a Klein bottle!

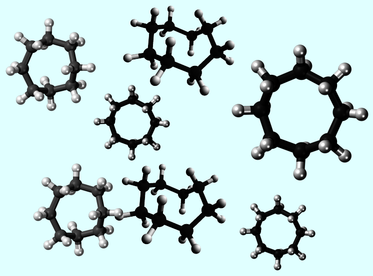
(and the authors actually show that the dataset concentrates near a Klein bottle embedded in $\mathbb{R}^9$.)

Conclusão

We can find interesting topology in datasets.







Invariants of homotopy classes allow to describe and understand them.

$$\beta_0(X) = 1, \quad \beta_1(X) = 2, \quad \beta_2(X) = 1$$

Thursday 28th: a stronger invariant, **homology**.

Tuesday 3rd: how to compute these invariants in practice? **persistent homology**.

Thursday 5th: **python tutorial** with the gudhi library.

A course about TDA: `https://raphaeltinarrage.github.io/EMAp.html`