Simplicial approximation to CW-complexes in practice

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Overview

Motivations. In combinatorial topology, it is common to represent a topological space X as a simplicial complex. This allows to:

- compute *topological invariants of* X (such as homology, fundamental groups),
- use *TDA techniques* that require a triangulation of X (such as persistent Stiefel-Whitney classes),
- study the *combinatorial complexity* of X (minimal triangulations, Lusternik–Schnirelmann category).

Surprisingly, there are a lot of important manifolds X for which we do not know explicit triangulations.

Besides, another well-studied representation of topological spaces is the notion of *CW complex*. We implement an algorithm that converts CW complexes into **homotopy equivalent** simplicial complexes.

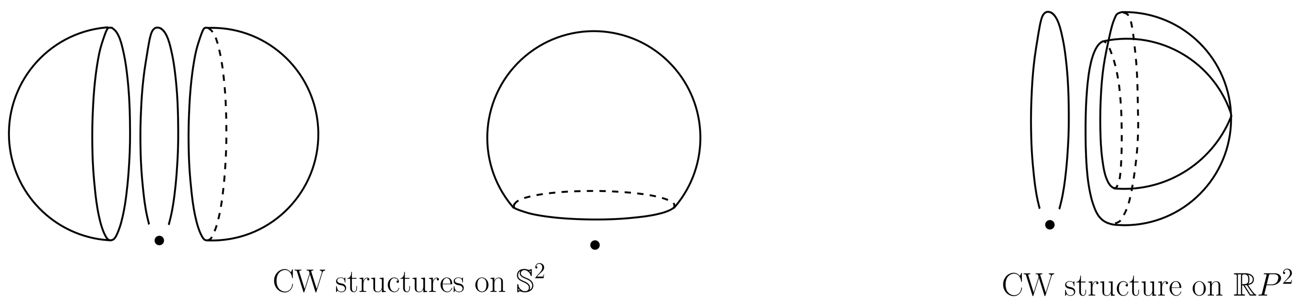
Notations

- $\mathcal{B}^n$ is the unit open ball of $\mathbb{R}^n$, $\overline{\mathcal{B}}^n$ its closed ball and $\mathbb{S}^{n-1}$ its sphere.
- K is a geometric simplicial complex, and $|K|$ its embedding.
- $\mathrm{St}(v)$ (resp. $\overline{\mathrm{St}}(v)$) is the open (resp. closed) star of a vertex $v \in K$.
- If $f\colon X \to Y$ is a continuous map, we denote its mapping cone by $\mathrm{Cone}(f) = (X \times [0,1] \sqcup Y) \,/\, ((x,1) \sim f(x) \text{ and } (x,0) \sim \text{point})$.

A **CW complex** is a topological space X with a partition $\{e_i \mid i \in [\![0,n]\!]\}$ such that:

- For each e_i, there exists an integer $d(i)$ and a homeomorphism $\Phi_i\colon \mathcal{B}^{d(i)} \to e_i$.
- This homeomorphism extends as a continuous map $\Phi_i\colon \overline{\mathcal{B}}^{d(i)} \to X$, called a *characteristic map*. We denote by $\overline{e}_i$ its image.

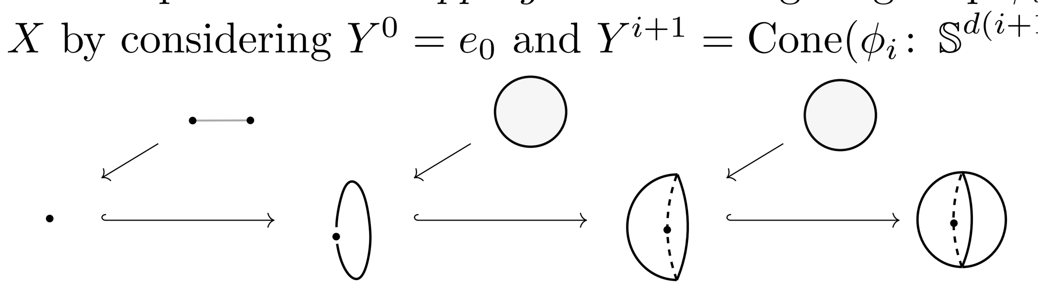
  The restriction of Φ_i to $\mathbb{S}^{d(i)-1}$, denoted ϕ_i, is called its *gluing map*.
- Each point $x \in \overline{e}_i \setminus e_i$ must lie in a cell e_j of lower dimension.



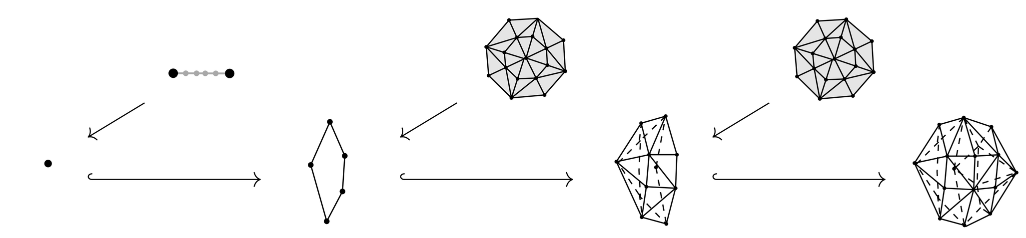
Any topological manifold of dimension $d \neq 4$ is homeomorphic to a CW complex.

Sketch of algorithm

Given a CW complex X and $i \le n$, we denote its i-skeleton as $X^i = \bigcup_{k\le i} e_k$.
Each X^i is homeomorphic to the *mapping cone* of the gluing map ϕ_i, $\mathrm{Cone}(\phi_i)$.
We can build X by considering $Y^0 = e_0$ and $Y^{i+1} = \mathrm{Cone}(\phi_i\colon \mathbb{S}^{d(i+1)-1} \to Y_i)$.



The algorithm consists in making this construction simplicial.

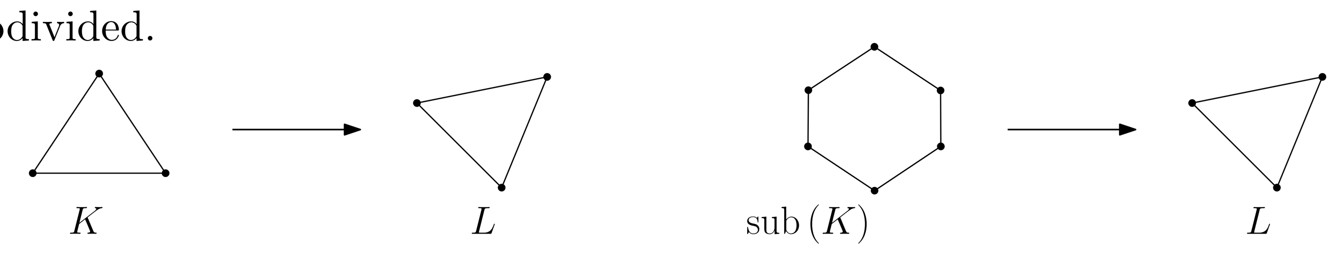


Lemma: $f, g\colon \mathbb{S}^d \to Y$ homotopic $\implies$ $\mathrm{Cone}(f)$ and $\mathrm{Cone}(g)$ homotopy equivalent.

Weak simplicial approximation

Consider two geometric simplicial complexes K, L and a map $f\colon |K| \to |L|$. The problem of **simplicial approximation** consists in finding a simplicial map $g\colon K \to L$ with geometric realization $g\colon |K| \to |L|$ homotopic to f.

The problem is easily solved when f satisfies the *star condition*. If not, K must be subdivided.



Star condition: For all vertex v of K, there exists a vertex w of L such that $f\left(|\overline{\mathrm{St}}(v)|\right) \subseteq |\mathrm{St}(w)|$.

Lemma: Any map g such that $f\left(|\overline{\mathrm{St}}(v)|\right) \subseteq |g(v)|$ is homotopic to f.

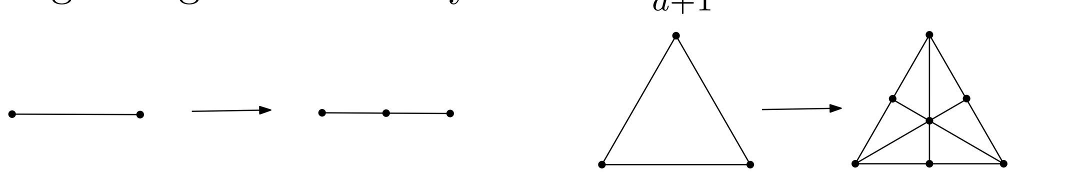
In practice, we will use the **weak star condition**: $f\left(|\overline{\mathrm{St}}(v)|\right) \subseteq |\mathrm{St}(w)|$.

Improvements can be implemented to simplify the problem:

- Before finding a simplicial approximation: *contract* the complex L.
- While finding it: only subdivise simplices where f does not satisfy the star condition.
- After finding it: with Delaunay subdivision, remove some vertices.

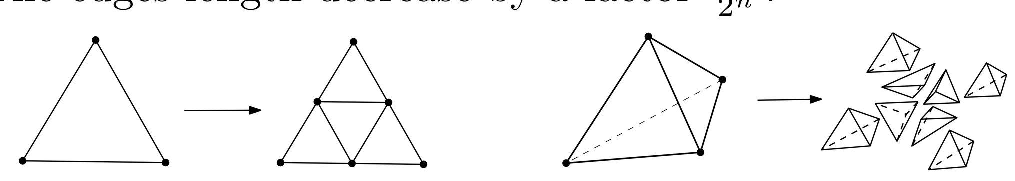
The **barycentric subdivision** of a d-simplex σ of $\mathbb{R}^n$, denoted $\mathrm{sub}(\sigma)$, consists in decomposing σ into a simplicial complex with $2^{d+1}-1$ vertices (the barycenters of its simplices) and $(d+1)!$ simplices of dimension d.

The edges length decrease by a factor $\frac{d}{d+1}$.

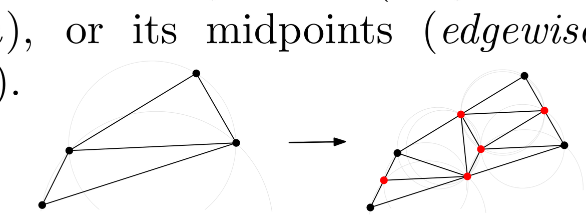


The **edgewise subdivisions** allows to decompose σ into a simplicial complex with only $\frac{d(d+1)}{2}$ vertices: the initial vertices, and the midpoints of its edges. It has 2^d simplices of dimension d.

The edges length decrease by a factor $\frac{\sqrt{d}}{2^n}$.

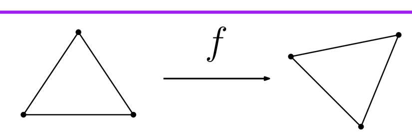


Delaunay subdivisions. A subset $X \subset \mathbb{S}^d$ induces a *Delaunay triangulation* of the sphere, denoted $\mathrm{Del}(X)$. In order to refine this complex, we can add vertices in X, for instance its barycenters (*barycentric Delaunay subdivision*), or its midpoints (*edgewise Delaunay subdivision*).

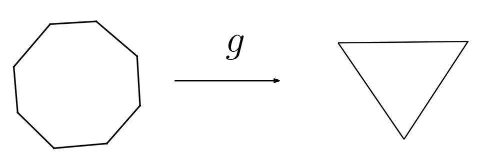


Triangulation of the mapping cone

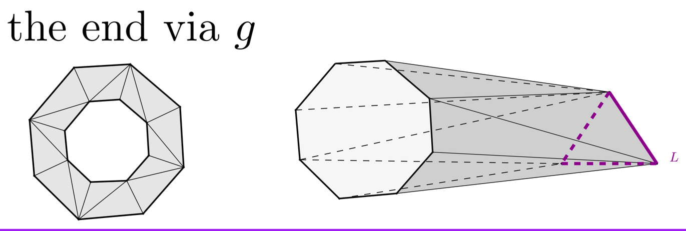
We want a **triangulation** $\mathrm{Cone}^{\mathrm{s}}(f)$ of $\mathrm{Cone}(f)$, the mapping cone of $f\colon |K| \to |L|$.



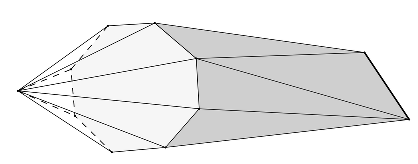
1. Find a simplicial approximation g to f



2. Triangulate $|K| \times [0,1]$ and glue L at the end via g



3. Cone the upper part of the cylinder



Checking homotopy: Consider $f, g\colon |K| \to |L|$.

Suppose that K and L are homeomorphic to $\mathbb{S}^d$. Then f and g are homotopic iff $H^d(\mathrm{Cone}(f)) \simeq H^d(\mathrm{Cone}(g))$. This allows to verify if a weak simplicial approximation is homotopic to the inital map.

Can this method be extended to more general spaces K, L?

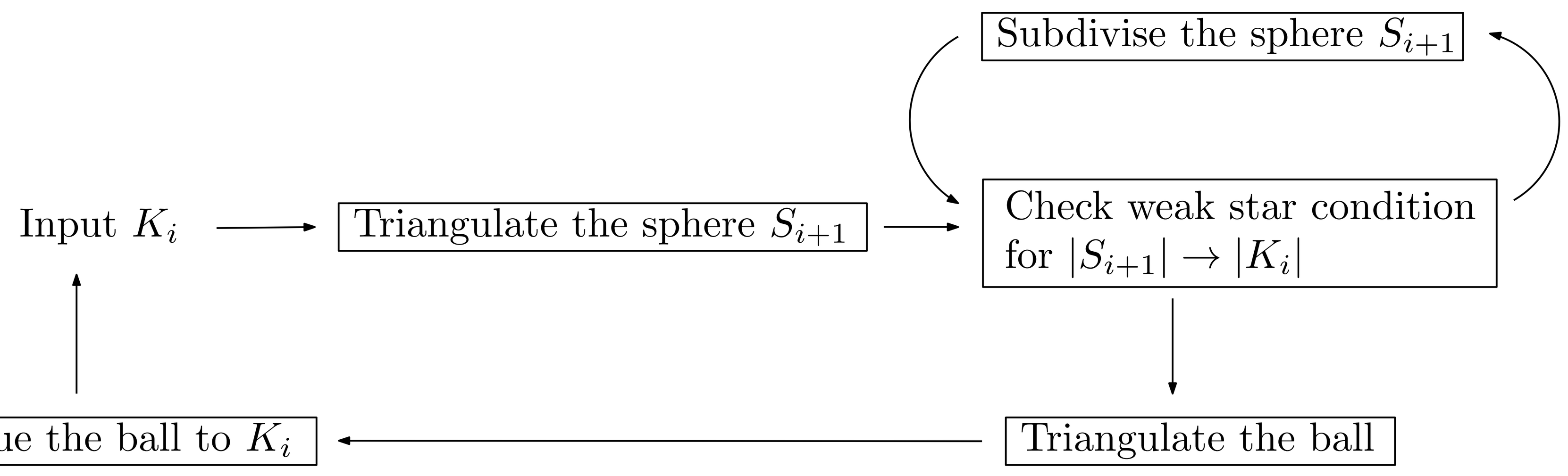
Algorithm

We build a sequence of simplicial complexes $K_0, \dots, K_n$ and, for each of them, a homotopy equivalence $h_i\colon X_i \to |K_i|$. Iteratively, we obtain K_{i+1} from K_i as follows:

- we find a triangulation $\iota_{i+1}\colon |S_{i+1}| \to \mathbb{S}^{d(i+1)-1}$ of the sphere such that the composition $h_i \circ \phi_{i+1} \circ \iota_{i+1}\colon |S_{i+1}| \to |K_i|$ satisfies the weak star condition,
- we choose a weak simplicial approximation $\phi'_{i+1}\colon S_{i+1} \to K_i$ to $h_i \circ \phi_{i+1} \circ \iota_{i+1}$,
- we define the simplicial mapping cone $K_{i+1} = \mathrm{Cone}^{\mathrm{s}}(\phi'_{i+1})$.

We obtain h_{i+1} as the following composition:

$$X_{i+1} \longrightarrow \mathrm{Cone}(\phi_{i+1}) \longrightarrow \mathrm{Cone}(\phi'_{i+1}) \longrightarrow |\mathrm{Cone}^{\mathrm{s}}(\phi'_{i+1})|$$



Theorem: Let X be a CW complex, and the algorithm on it. Suppose that each weak simplicial approximation ϕ'_i computed by the algorithm is homotopy equivalent to ϕ_i.

Correctness: If the algorithm terminates, then it returns a simplicial complex homotopy equivalent to X.

Termination: The termination depends on the algorithm used for the subdivision loop, the subdivision method, and the dimension d, as described in the following table (a cross indicates that the algorithm does not terminate in general).

	global subdivisions	generalized subdivisions
Barycentric	any d	×
Edgewise	any d	×
Delaunay barycentric	$d \le 4$	×
Delaunay edgewise	$d \le 3$	×

Applications. We apply the algorithm on a few CW complexes. We give the number of vertices of the output complexes (and in parenthesis the number of vertices before contraction).

Projective spaces	$\mathbb{R}P^1$	$\mathbb{R}P^2$	$\mathbb{R}P^3$
Barycentric	3 (4)	6 (40)	1240 (13'179)
Edgewise	3 (4)	7 (40)	86 (2443)
Delaunay barycentric	3 (4)	6 (11)	14 (77)
Delaunay edgewise	3 (4)	6 (11)	11 (92)

Lens spaces $L(p,q)$

$q \backslash p$	2	3	4	5	6	7
1	12	17	36	58	63	121
2		20		50		106
3	12		34	51		96
4		16		76		104
5	12	14	38		70	107

Grassmannian $\mathcal{G}(2,\mathbb{R}^4)$ of planes in $\mathbb{R}^4$: 1691 (6092) vertices