EMAp Summer Course

Topological Data Analysis with Persistent Homology

https://raphaeltinarrage.github.io/EMAp.html

Lesson 9: Decomposition of persistence modules

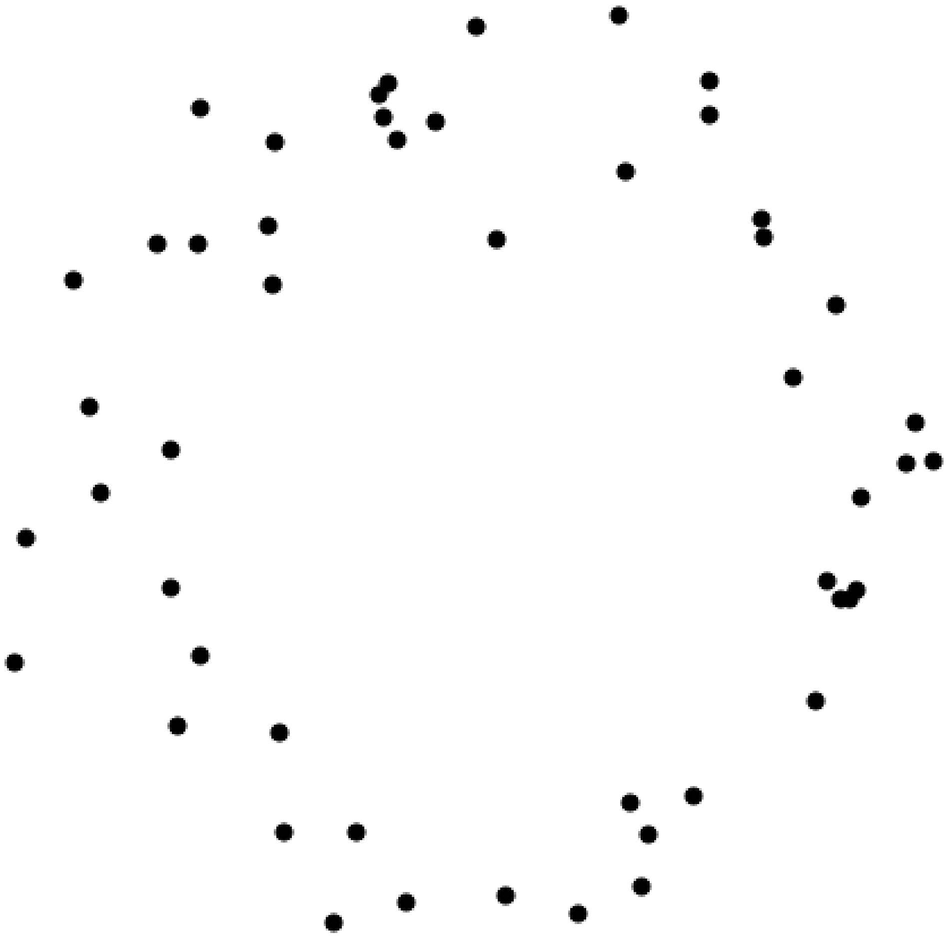
Last update: February 8, 2021

Introduction

Let $X \subset \mathbb{R}^n$ finite.

Pipeline of homology inference:

- select a thickening X^t
- compute its homology via $\check{\mathrm{C}}\mathrm{ech}^t(X)$ or $\mathrm{Rips}^t(X)$



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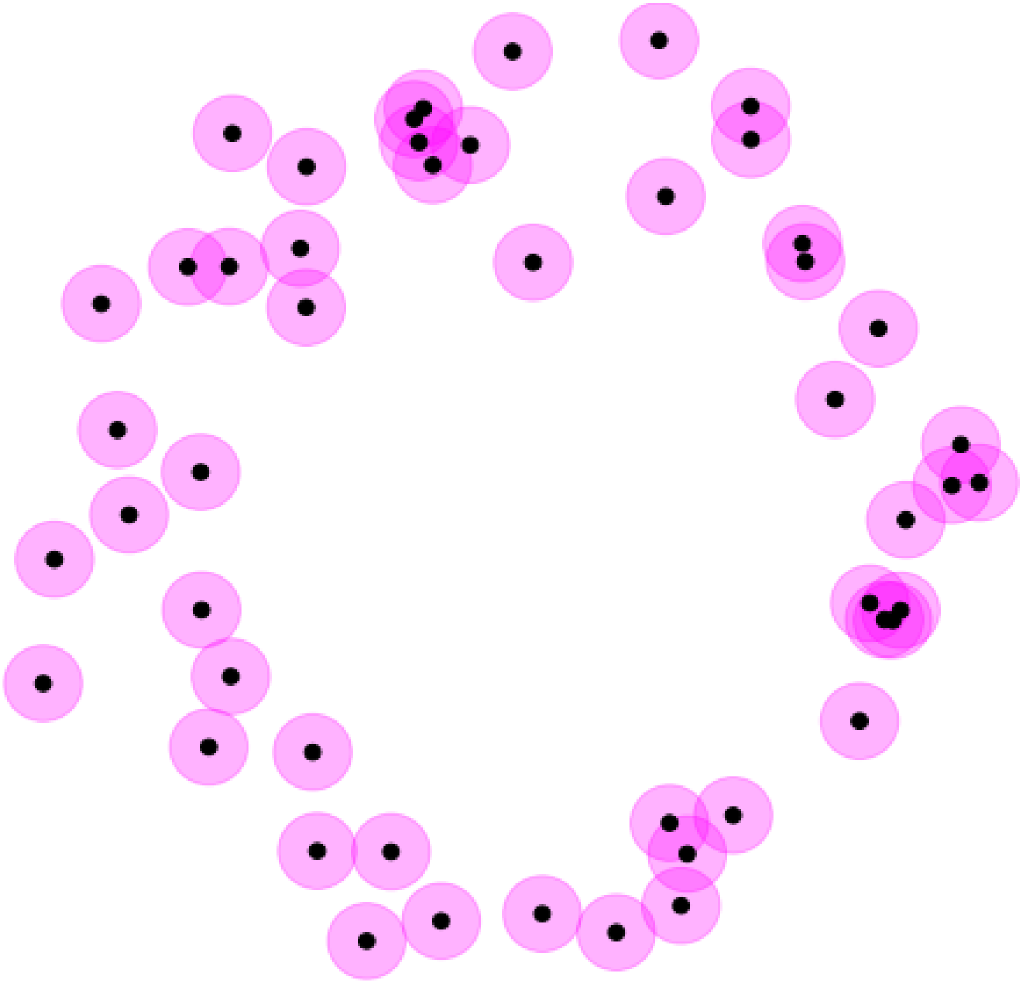

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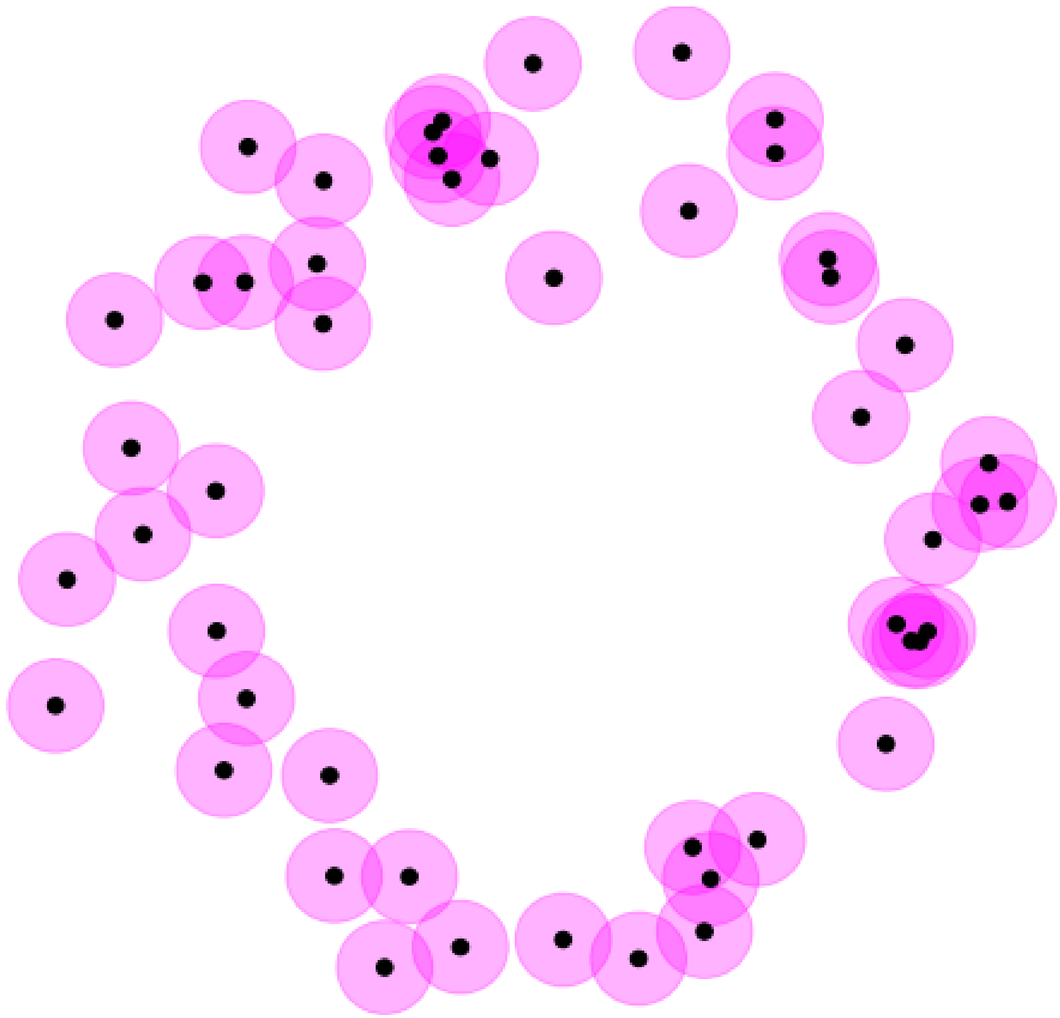

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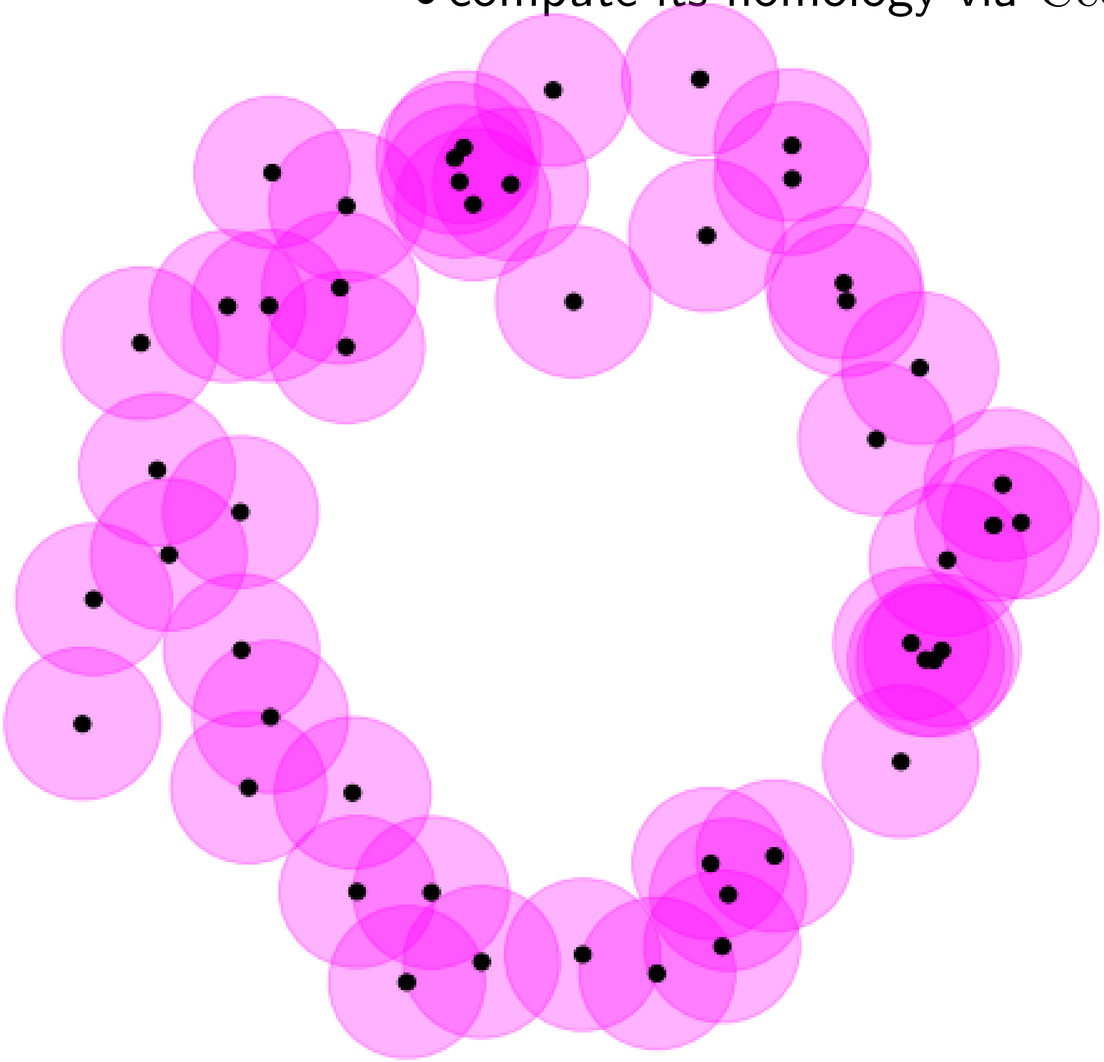

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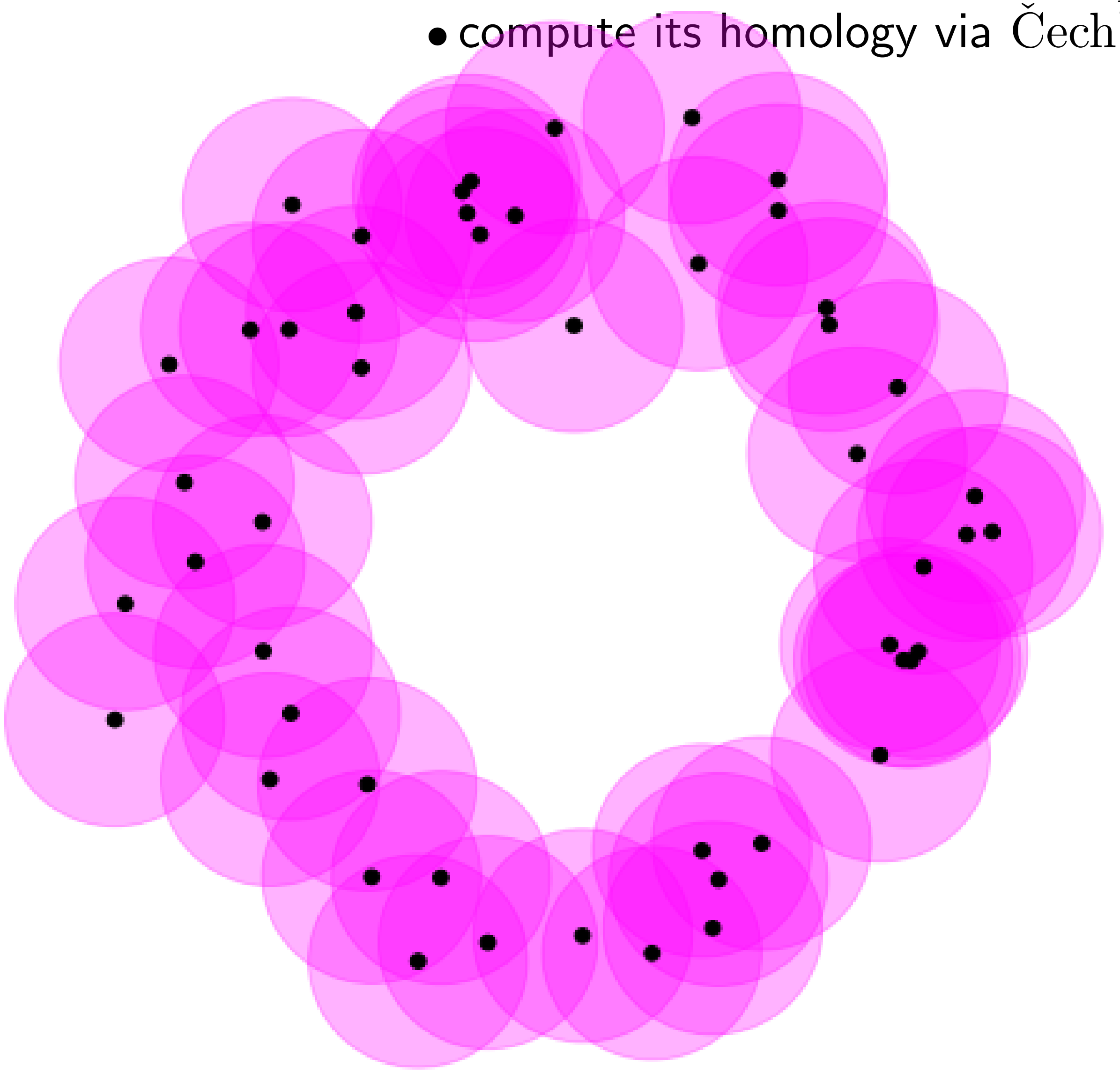

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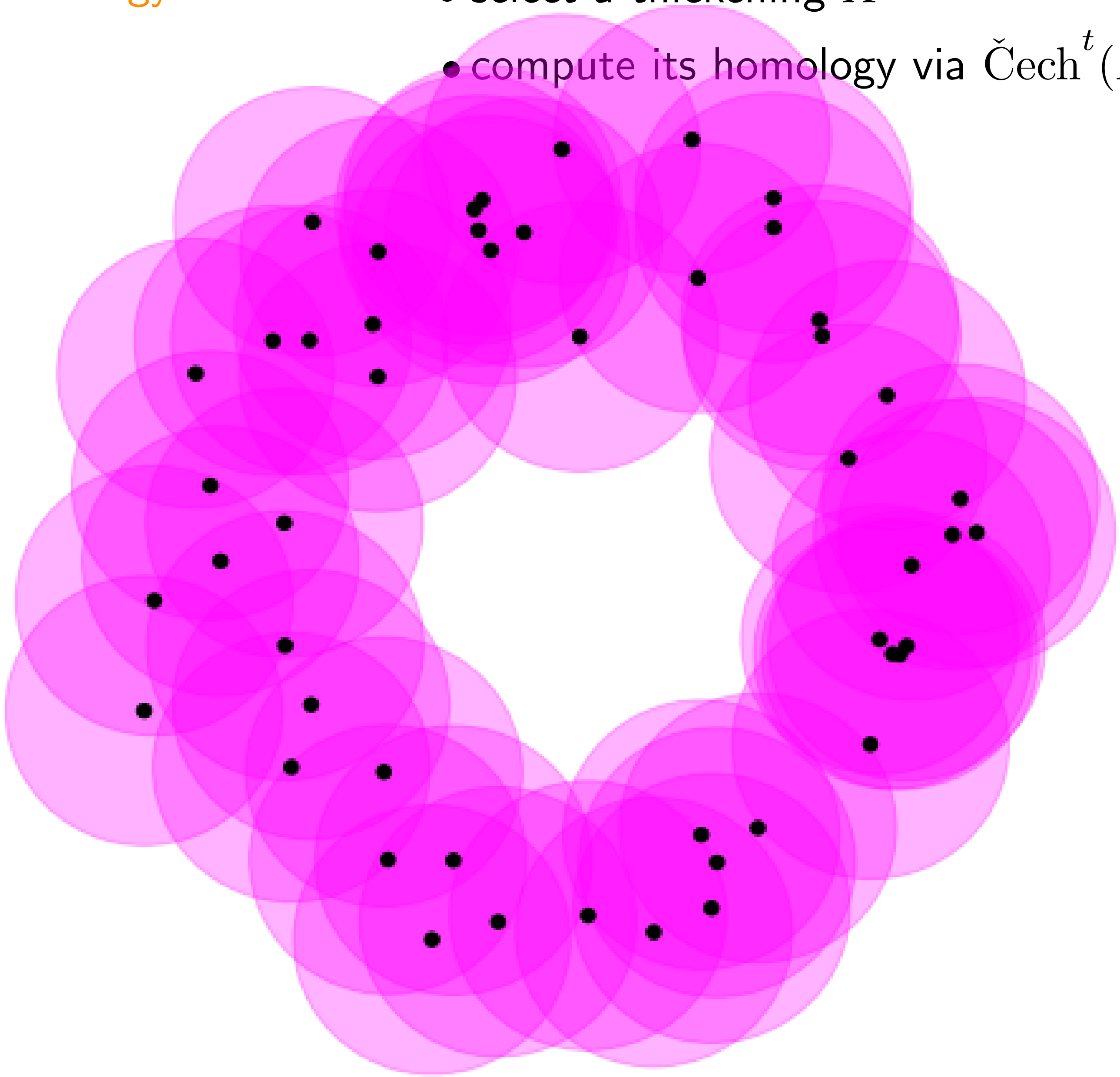

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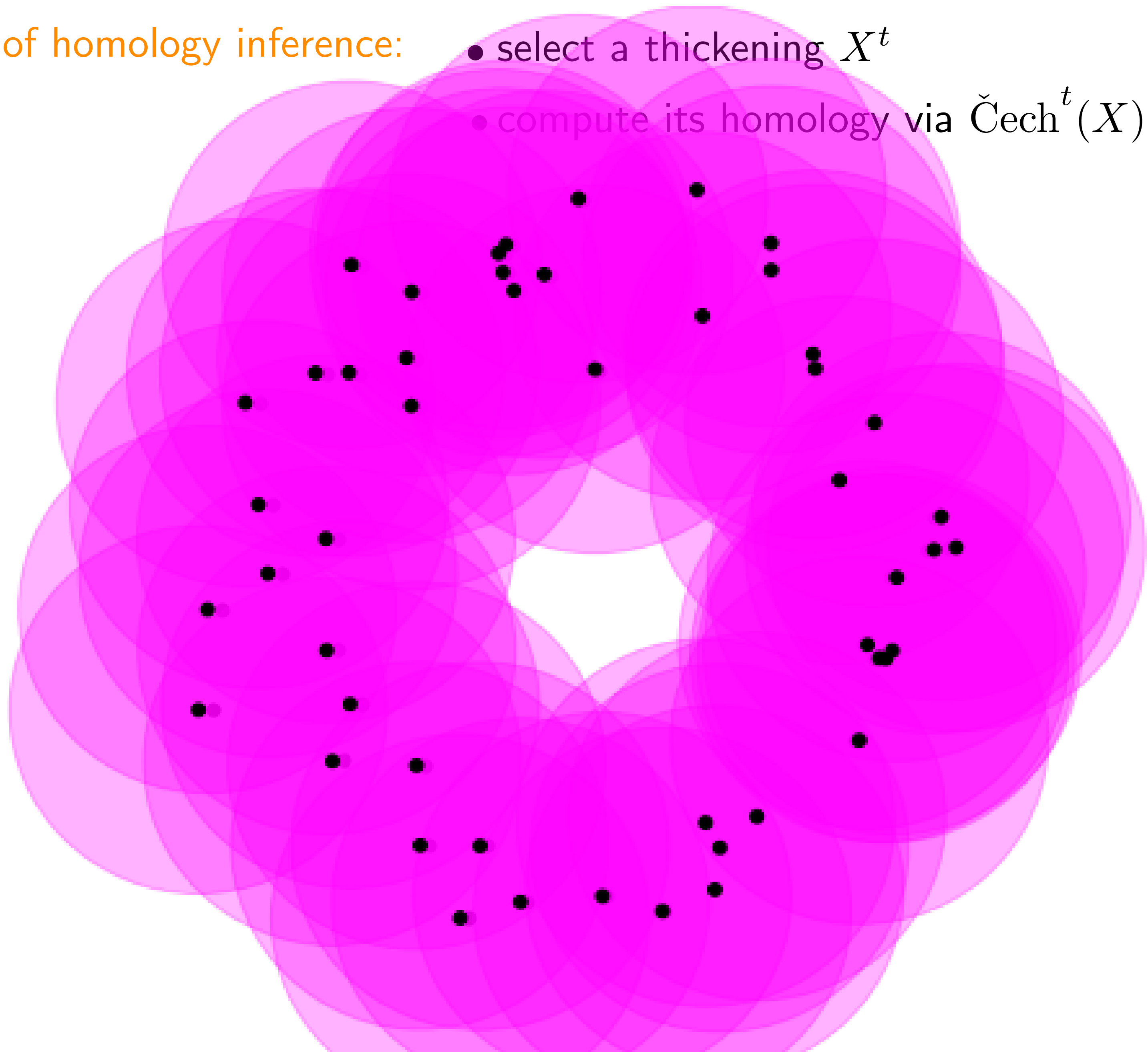

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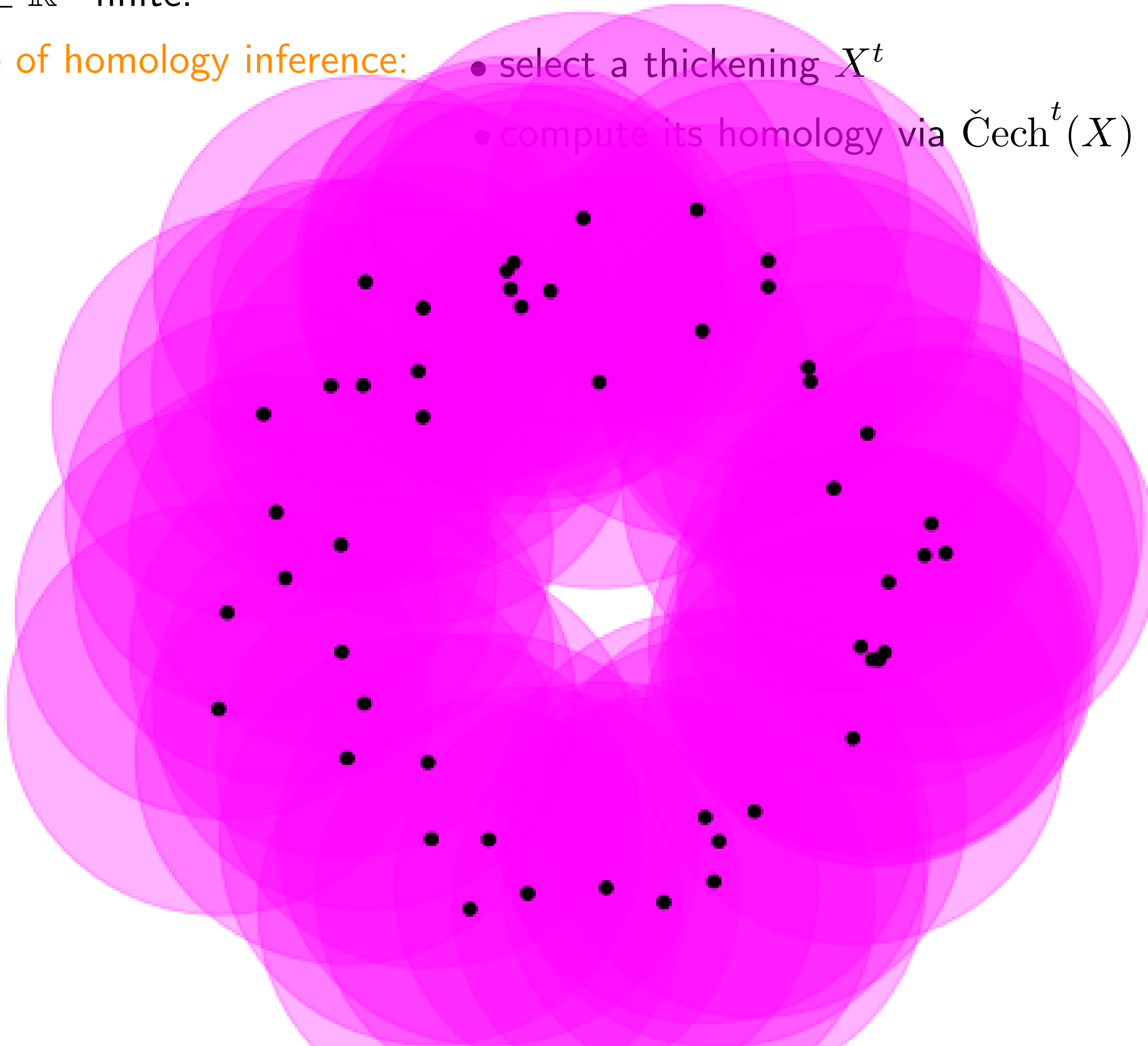

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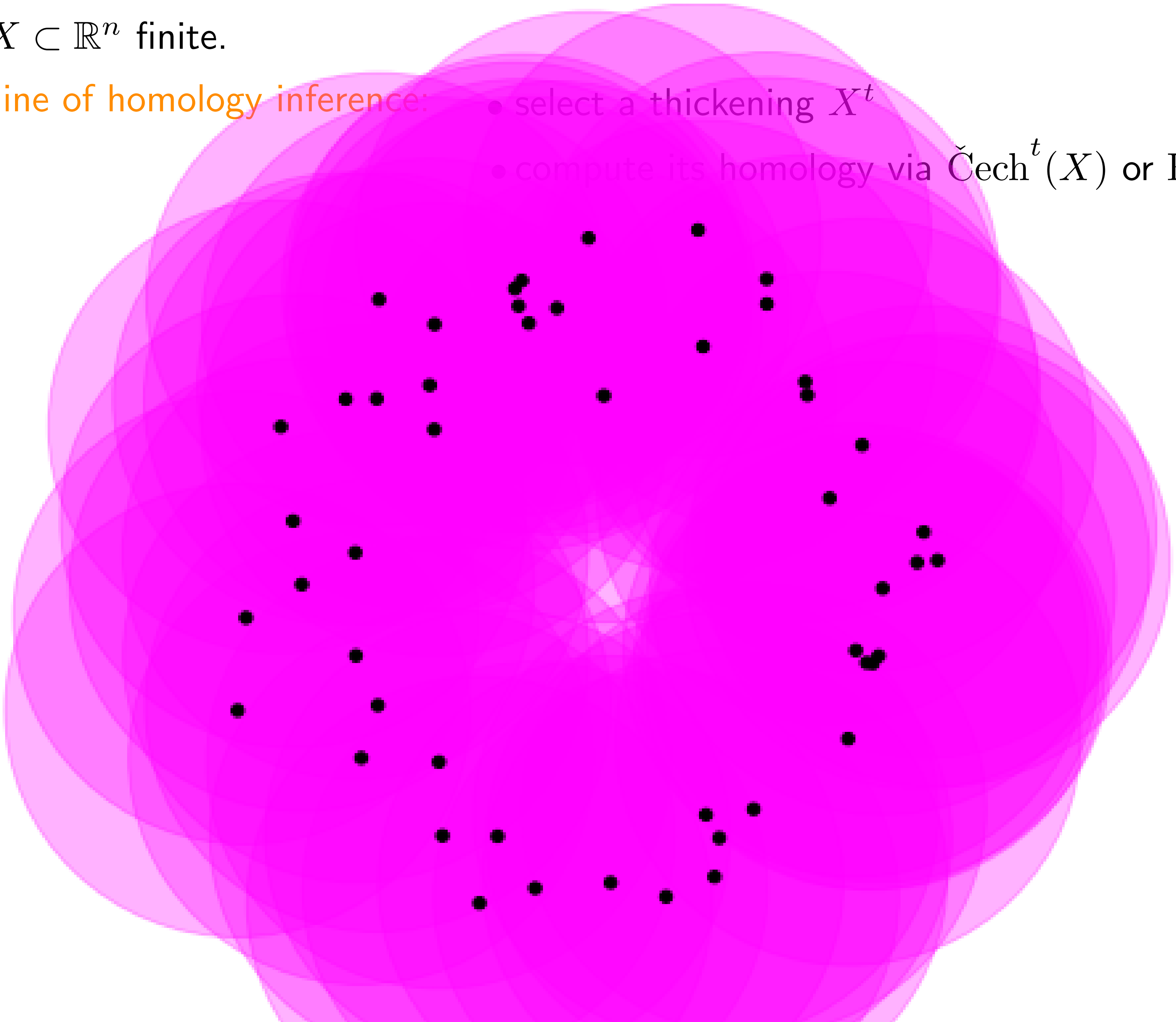

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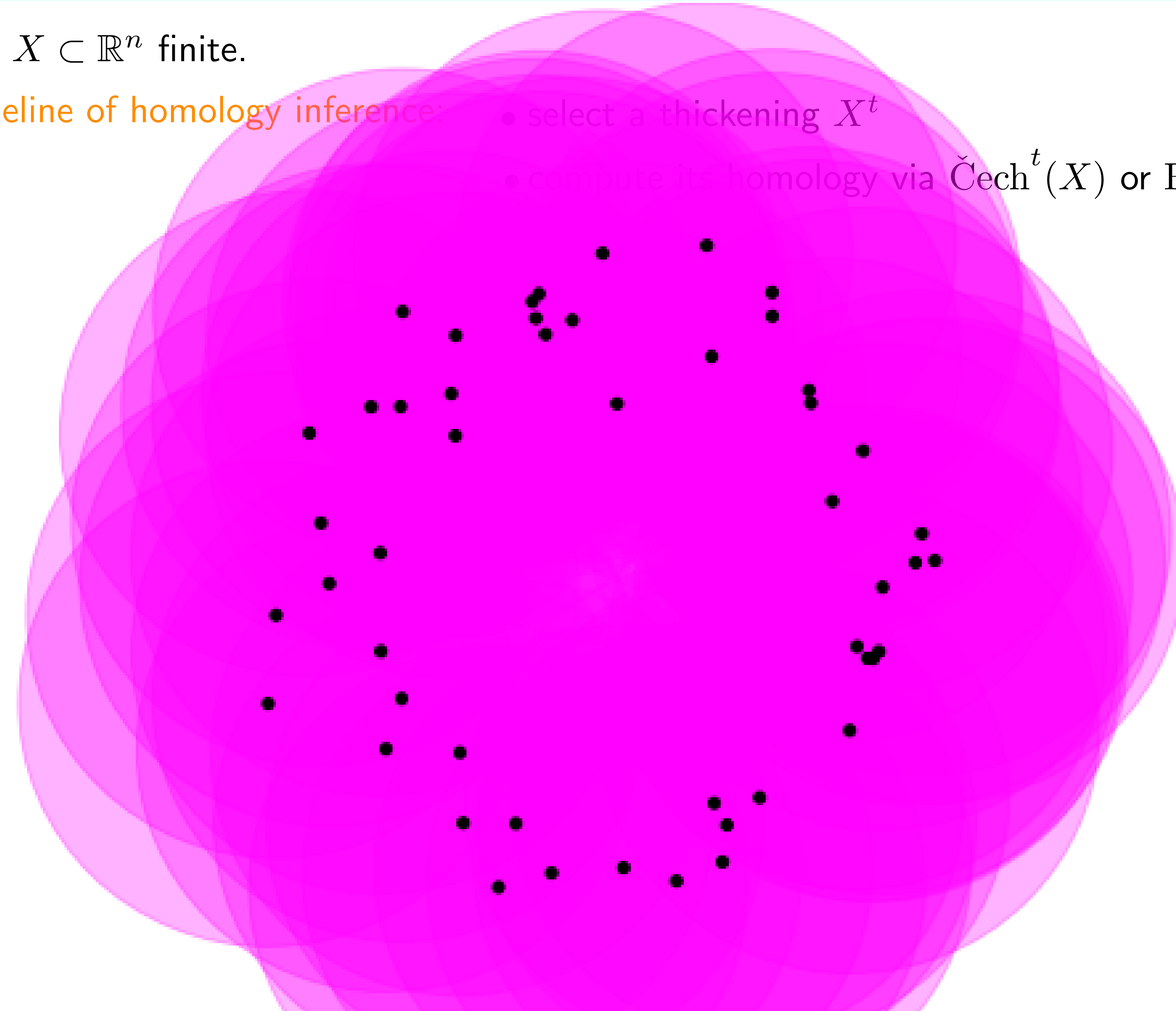

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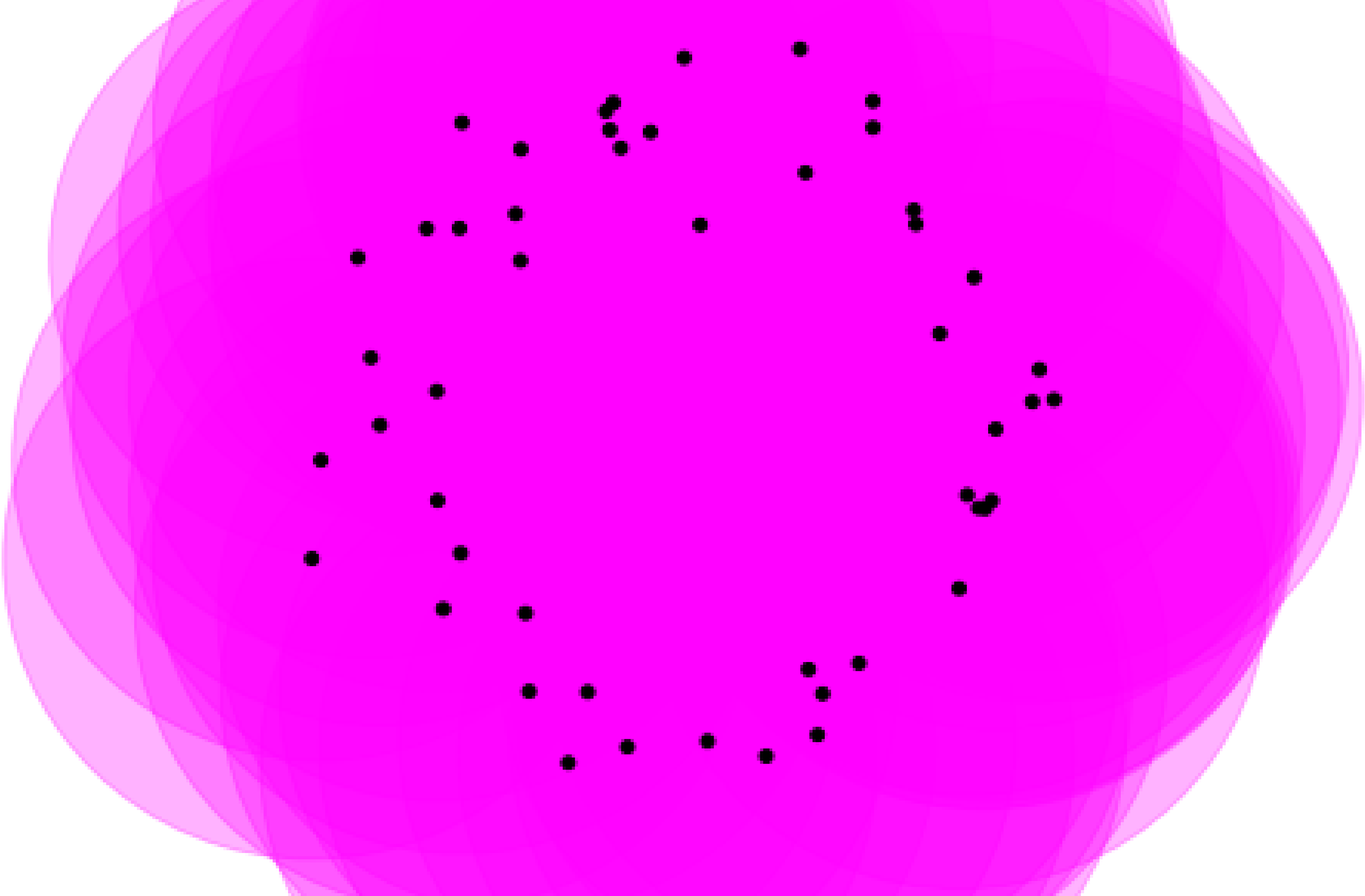

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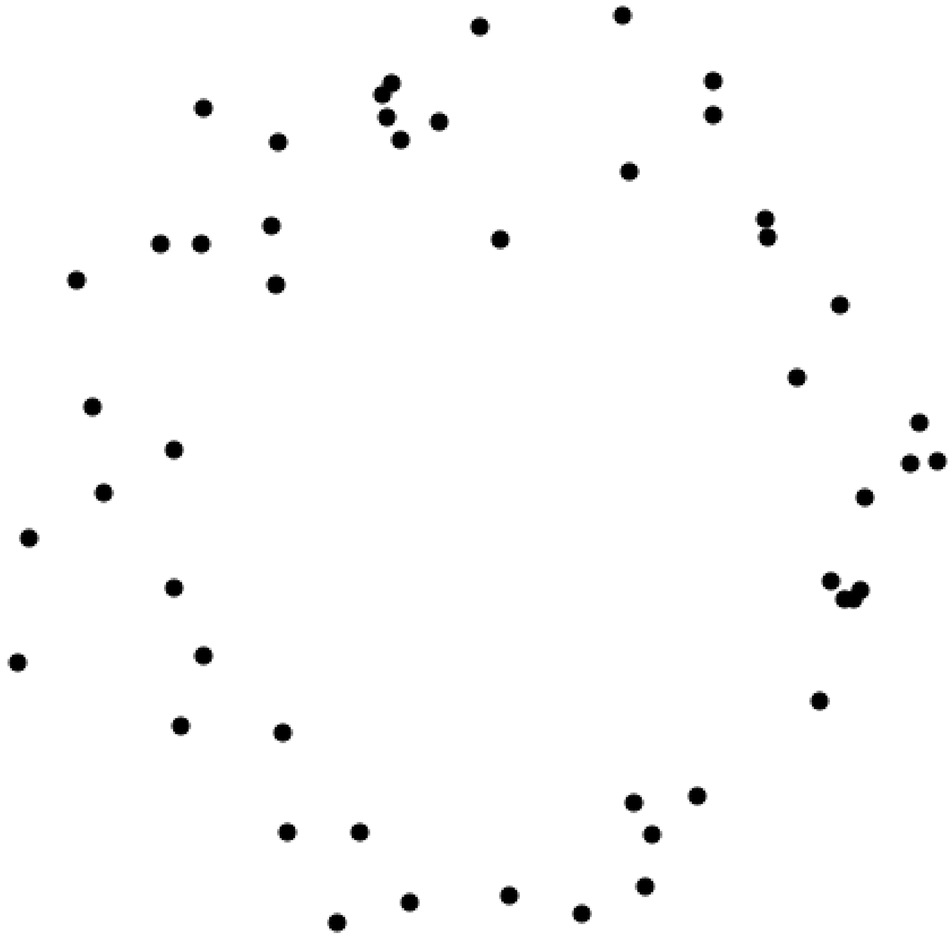

How to handle topological noise?

I - Functoriality of homology

II - Persistence modules

III - Decomposition

IV - Persistence algorithm

Homology is a functor

We have seen that homology transforms topological spaces into vector spaces

$$\begin{aligned} H_i \colon \mathrm{Top} &\longrightarrow \mathrm{Vect} \\ X &\longmapsto H_i(X) \end{aligned}$$

Actually, it also transforms *continous maps* into *linear maps*

$$\big(f \colon X \to Y\big) \longmapsto \big(f_* \colon H_i(X) \to H_i(Y)\big)$$

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We will adopt a simplicial point of view.

$$\begin{aligned} H_i \colon \mathrm{SimpComp} &\longrightarrow \mathrm{Vect} \\ K &\longmapsto H_i(K) \\ \big(f \colon K \to L\big) &\longmapsto \big(f_* \colon H_i(K) \to H_i(L)\big) \end{aligned}$$

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what is a map between simplicial complexes?

Simplicial maps

Definition: Let K and L be two simplicial complexes, and V_K, V_L their set of vertices. A *simplicial map* between K and L is a map $f\colon V_K \to V_L$ such that

$$\forall \sigma \in K, f(\sigma) \in L.$$

When there is no risk of confusion, we may denote a simplicial map $f\colon K \to L$ instead of $f\colon V_K \to V_L$.

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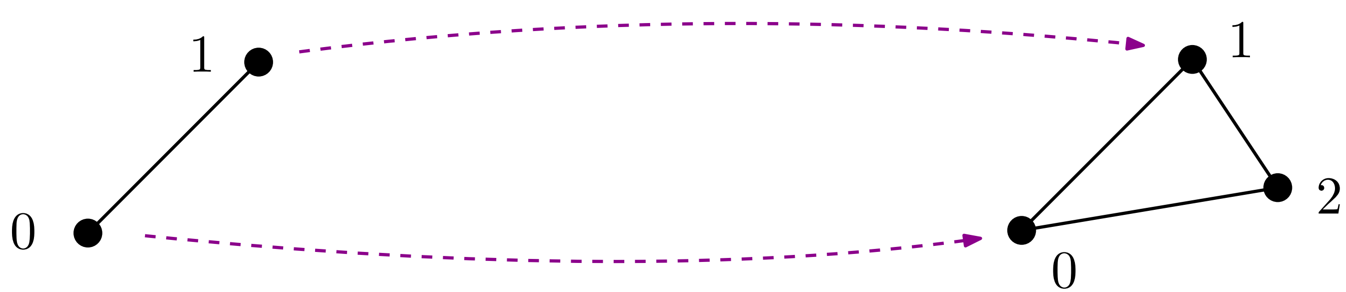
Example: Let $K = \{[0],[1],[0,1]\}$, $L = \{[0],[1],[2],[0,1],[0,2],[1,2]\}$ and

$$f\colon \{0,1\} \to \{0,1,2\}$$
$$0 \mapsto 0$$
$$1 \mapsto 1$$



It is simplicial since $f([0,1]) = [0,1]$ is a simplex of L.

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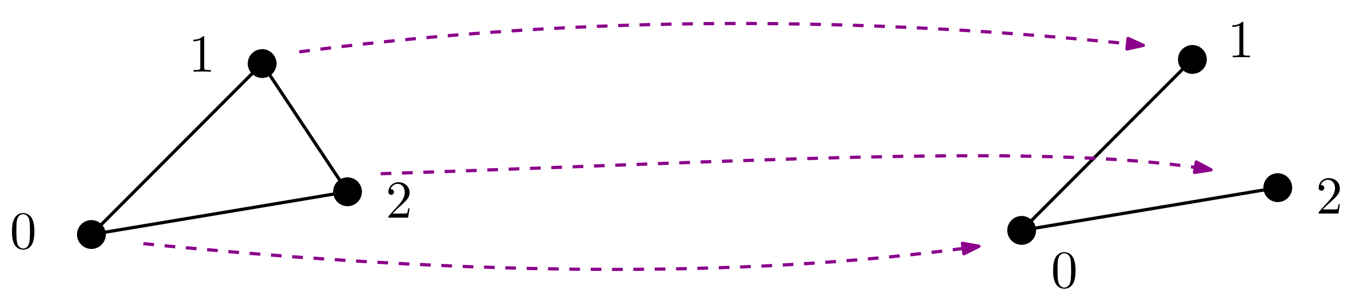
$$0 \mapsto 0$$
$$1 \mapsto 1$$
$$2 \mapsto 2$$



It is not simplicial since $f([1,2]) = [1,2]$ is not a simplex of L.

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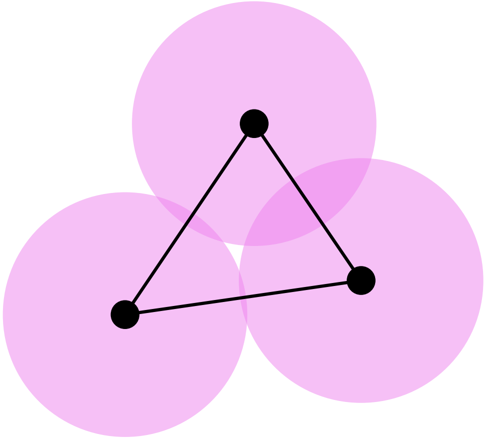
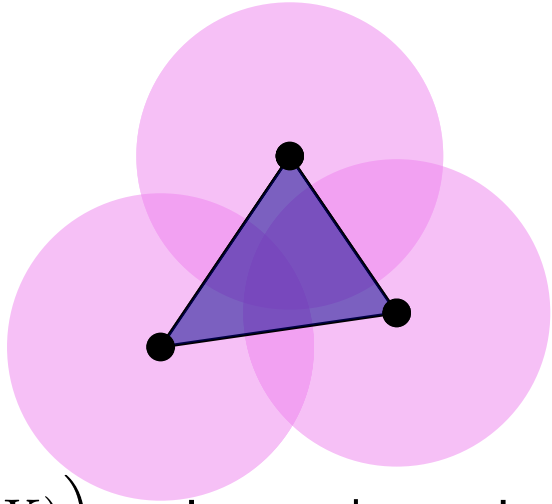
Example: Let $X \subset \mathbb{R}^n$ and $s, t \geq 0$ such that $s \leq t$. Consider the Čech complexes $\check{\mathrm{C}}\mathrm{ech}^s(X)$ and $\check{\mathrm{C}}\mathrm{ech}^t(X)$.

The inclusion map $i\colon \check{\mathrm{C}}\mathrm{ech}^s(X) \to \check{\mathrm{C}}\mathrm{ech}^t(X)$ is a simplicial map.





Indeed, the sequence of simplicial complexes $\left(\check{\mathrm{C}}\mathrm{ech}^t(X)\right)_{t\geq 0}$ is non-decreasing.

Induced map

Let $f\colon K \to L$ be a simplicial map. Let $n \geq 0$, and consider the groups of chains of K and L:

$$C_n(K) = \left\{ \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma,\ \forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z} \right\}$$

$$C_n(L) = \left\{ \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma,\ \forall \sigma \in L_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z} \right\}$$

We define a linear map as follows:

$$\begin{aligned} f_n\colon C_n(K) &\longrightarrow C_n(L) \\ \sigma &\longmapsto f(\sigma) \quad \text{if} \quad \dim(f(\sigma)) = n, \\ & \qquad 0 \quad \text{else.} \end{aligned}$$

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\end{array}
$$

Lemma: For every $n \geq 0$, we have $\partial_n \circ f_n = f_{n-1} \circ \partial_n$.

Proof: Let $\sigma \in K_{(n)}$. We have the equalities

$$
\partial_n \circ f_n(\sigma) = \sum_{\substack{\mu \subset f(\sigma) \\ |\mu| = |\sigma| - 1}} \mu
$$

$$
f_{n-1} \circ \partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} f_n(\tau)
$$

We should distinguish three cases:

- $|f(\sigma)| = |\sigma|$ (i.e. f is injective on σ),
- $|f(\sigma)| < |\sigma| - 1$,
- $|f(\sigma)| = |\sigma| - 1$.

Induced map

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Lemma: For every $n \geq 0$, we have $\partial_n \circ f_n = f_{n-1} \circ \partial_n$.

Proposition: For every $c \in Z_n(K)$, we have $f_n(c) \in Z_n(L)$.
For every $c \in B_n(K)$, we also have $f_n(c) \in B_n(L)$.

Proof: First, let $c \in Z_n(K)$. We have

$$\partial_n \circ f_n(c) = f_{n-1} \circ \partial_n(c) = f_{n-1}(0) = 0,$$

hence $f_n(c) \in Z_n(L)$.

Secondly, let $c \in B_n(K)$, and write $c = \partial_{n+1}(c')$ with $c' \in C_{n+1}(K)$. We get

$$f_n(c) = f_n \circ \partial_{n+1}(c') = \partial_{n+1} \circ f_{n+1}(c'),$$

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We have $B_n(K) \subset Z_n(K)$, $B_n(L) \subset Z_n(L)$, $f(Z_n(K)) \subset f(Z_n(K))$ and $f(B_n(K)) \subset f(B_n(K))$.
Hence we can define a linear map between quotient vector spaces:

$$(f_n)_*\colon Z_n(K)/B_n(K) \longrightarrow Z_n(L)/B_n(L).$$

By definition of the homology groups, we have defined a map

$$(f_n)_*\colon H_n(K) \longrightarrow H_n(L).$$

It is called the *induced map in homology*.

Induced map

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$$
\begin{array}{ccccc}
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 & \downarrow{\scriptstyle (f_3)_*} & \downarrow{\scriptstyle (f_2)_*} & \downarrow{\scriptstyle (f_1)_*} & \downarrow{\scriptstyle (f_0)_*} \\
\cdots & H_3(L) & H_2(L) & H_1(L) & H_0(L)
\end{array}
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$(f_n)_*$ can be defined as

$$
c = \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \longmapsto \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot f_n(\sigma)
$$

Induced map

Example: Consider the simplicial complexes $K = L = \{[0], [1], [2], [0, 1], [0, 2], [1, 2]\}$.

The inclusion $i\colon K \to L$ induces the identity in H^0:

$$(i_1)_*\colon H_0(K) \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow H_0(L) \simeq \mathbb{Z}/2\mathbb{Z}$$
$$1 \longmapsto 1$$

The inclusion $i\colon K \to L$ induces the identity in H^1:

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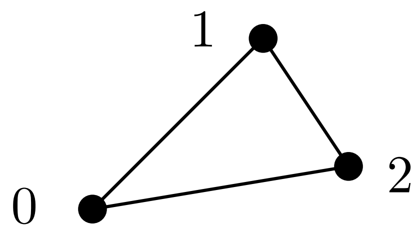

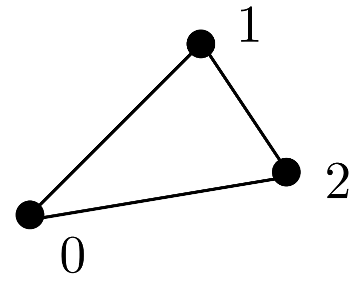

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Induced map

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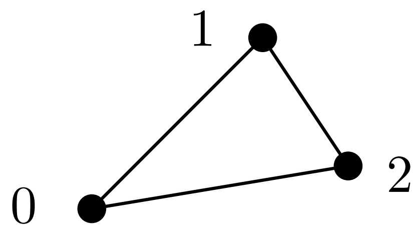
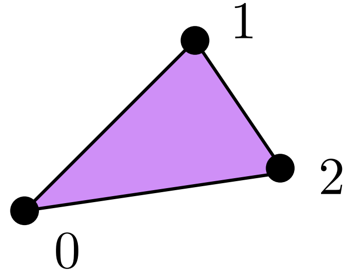
The inclusion $i\colon K \to L$ induces the zero map in H^1:

$$(i_1)_*\colon H_1(K) \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow H_1(L) \simeq \{0\}$$
$$1 \longmapsto 0$$





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Induced map

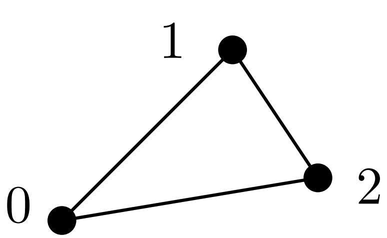
Example: Consider the simplicial complexes $K = \{[0],[1],[2],[0,1],[0,2],[1,2]\}$ and $L = \{[0],[1],[2],[3],[0,1],[0,2],[1,2],[1,3],[2,3]\}$.

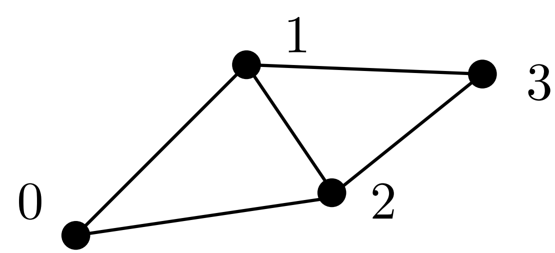
The homology group $H_1(L)$ is isomorphic to the vector space $(\mathbb{Z}/2\mathbb{Z})^2$ by identifying $[0,1]+[0,2]+[1,2] \mapsto (1,0)$ and $[1,2]+[2,3]+[1,3] \mapsto (0,1)$.

The inclusion $i\colon K \to L$ induces the following map between 1^{st} homology groups:

$$\begin{aligned}(i_1)_*\colon H_1(K) \simeq \mathbb{Z}/2\mathbb{Z} &\longrightarrow H_1(L) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \\ 1 &\longmapsto (1,0)\end{aligned}$$

It can be represented as the matrix $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.





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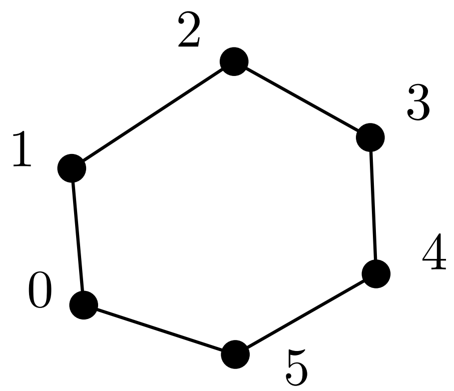
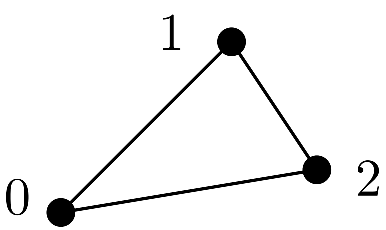
Induced map

Exercise: Let $K = \{[0],[1],[2],[3],[4],[5],[0,1],[1,2],[2,3],[3,4],[4,5],[5,0]\}$ and $L = \{[0],[1],[2],[0,1],[1,2],[2,0]\}$.

Consider the simplical map $f\colon i \mapsto i$ modulo 3.

Show that the induced map $(f_1)_*$ is zero.





$(f_n)_*$ can be defined as

$$c = \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \longmapsto \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot f_n(\sigma)$$

Functor property

Proposition: Let K, L, M be three simplicial complexes, and consider two simplicial maps $f\colon K \to L$ and $g\colon L \to M$.

For any $n \geq 0$, the induced map $((g \circ f)_n)_*\colon H_n(K) \to H_n(M)$ and $(g_n)_* \circ (f_n)_*\colon H_n(K) \to H_n(M)$ are equal.

$$K \xrightarrow[f]{} L \xrightarrow[g]{} M, \quad K \xrightarrow{g \circ f} M$$

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Proof: Let $\sigma \in K_{(n)}$. The image $(g \circ f)_n(\sigma)$ is

- $(g \circ f)(\sigma)$ if $g \circ f$ is injective on σ,
- 0 else.

If $g \circ f$ is injective on σ, then f is injective on σ **and** g is injective on $f(\sigma)$, hence $g_n \circ f_n(\sigma) = g \circ f(\sigma)$, and we deduce the result.

If $g \circ f$ is not injective on σ, then f is not injective on σ **or** g is not injective on $f(\sigma)$, hence $g_n \circ f_n(\sigma) = 0$, and we deduce the result.

I - Functoriality of homology

II - Persistence modules

III - Decomposition

IV - Persistence algorithm

Tracking cycles over time

Let $X \subset \mathbb{R}^n$. The collection of its thickenings is an non-decreasing sequence of subsets

$$\ldots \subset X^{t_1} \subset X^{t_2} \subset X^{t_3} \subset \ldots$$

By considering the corresponding Čech complexes, we obtain an non-decreasing sequence of simplicial complexes

$$\ldots \subset \check{\mathrm{C}}\mathrm{ech}^{t_1}(X) \subset \check{\mathrm{C}}\mathrm{ech}^{t_2}(X) \subset \check{\mathrm{C}}\mathrm{ech}^{t_3}(X) \subset \ldots$$

Let us denote i_s^t the inclusion map corresponding to $\check{\mathrm{C}}\mathrm{ech}^s(X) \subset \check{\mathrm{C}}\mathrm{ech}^t(X)$. We can write

$$\dashrightarrow \check{\mathrm{C}}\mathrm{ech}^{t_1}(X) \xrightarrow{i_{t_1}^{t_2}} \check{\mathrm{C}}\mathrm{ech}^{t_2}(X) \xrightarrow{i_{t_2}^{t_3}} \check{\mathrm{C}}\mathrm{ech}^{t_3}(X) \dashrightarrow$$

Applying the i^{th} homology functor yields a diagram of vector spaces

$$\dashrightarrow H_i(\check{\mathrm{C}}\mathrm{ech}^{t_1}(X)) \xrightarrow{(i_{t_1}^{t_2})_*} H_i(\check{\mathrm{C}}\mathrm{ech}^{t_2}(X)) \xrightarrow{(i_{t_2}^{t_3})_*} H_i(\check{\mathrm{C}}\mathrm{ech}^{t_3}(X)) \dashrightarrow$$

where the maps $(i_s^t)_*$ are those induced in homology by the inclusions i_s^t.

Tracking cycles over time

$$\dashrightarrow H_i(\check{\text{C}}\text{ech}^{t_1}(X)) \xrightarrow{\left(i_{t_1}^{t_2}\right)_*} H_i(\check{\text{C}}\text{ech}^{t_2}(X)) \xrightarrow{\left(i_{t_2}^{t_3}\right)_*} H_i(\check{\text{C}}\text{ech}^{t_3}(X)) \dashrightarrow$$

Let $i \geq 0$, $t_0 \geq 0$ and consider a cycle $c \in H_i(\check{\text{C}}\text{ech}^{t_0}(X))$.

Its *death time* is: $\sup\left\{t \geq t_0, \left(i_{t_0}^{t}\right)(c) \neq 0\right\}$,

its *birth time* is: $\inf\left\{t \geq t_0, \left(i_{t}^{t_0}\right)^{-1}(\{c\}) \neq \emptyset\right\}$,

its *persistence* is the difference.

As a rule of thumb:

- cycles with large persistence correspond to important topological features of the dataset,
- cycles with short persistence corresponds to topological noise.

Tracking cycles over time

$$\dashrightarrow H_i(\check{\mathrm{C}}\mathrm{ech}^{t_1}(X)) \xrightarrow{(i_{t_1}^{t_2})_*} H_i(\check{\mathrm{C}}\mathrm{ech}^{t_2}(X)) \xrightarrow{(i_{t_2}^{t_3})_*} H_i(\check{\mathrm{C}}\mathrm{ech}^{t_3}(X)) \dashrightarrow$$

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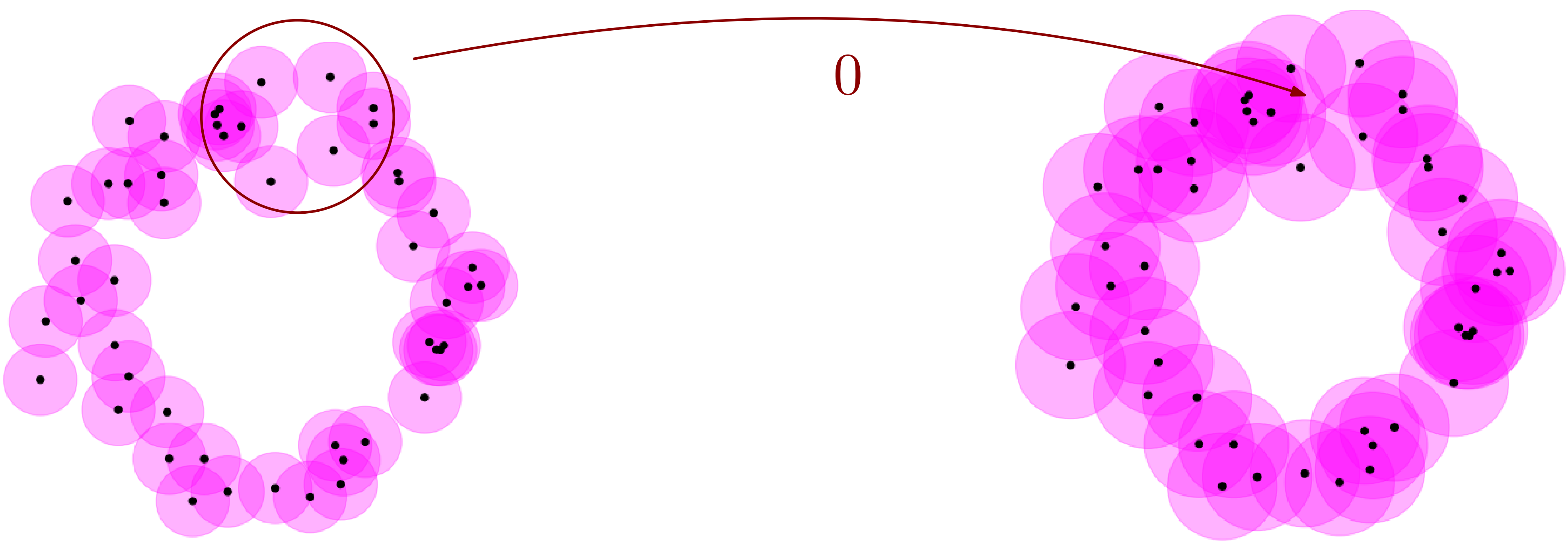

Tracking cycles over time

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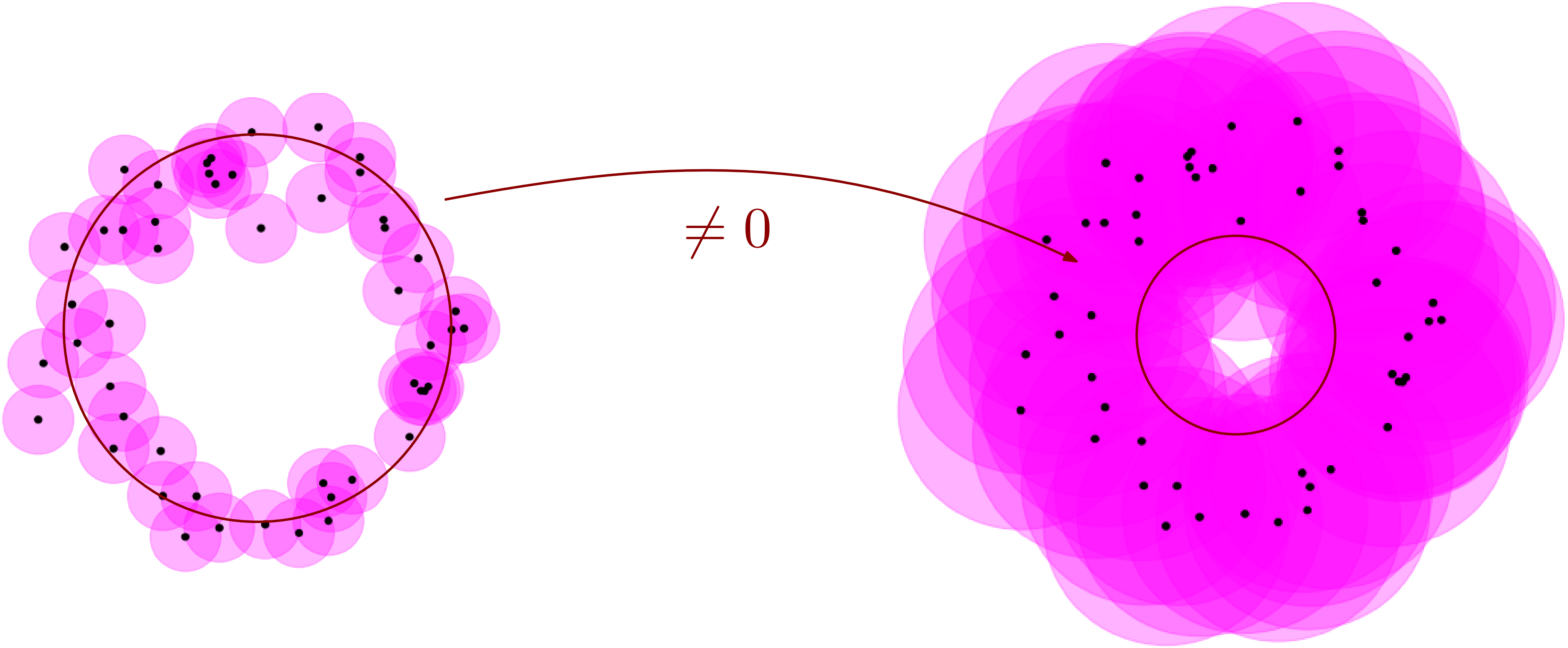

Persistence modules (finalmente!)

Definition: A *persistence module* $\mathbb{V}$ *over* $\mathbb{R}^+$ *with coefficients in* $\mathbb{Z}/2\mathbb{Z}$ is a pair $(\mathbb{V}, v)$ where $\mathbb{V} = (V^t)_{t\in\mathbb{R}^+}$ is a family of $\mathbb{Z}/2\mathbb{Z}$-vector spaces, and $v = (v_s^t \colon V^s \to V^t)_{s\leq t\in\mathbb{R}^+}$ a family of linear maps such that:

- for every $t \in \mathbb{R}^+$, $v_t^t \colon V^t \to V^t$ is the identity map,
- for every $r, s, t \in \mathbb{R}^+$ such that $r \leq s \leq t$, we have $v_s^t \circ v_r^s = v_r^t$.

When the context is clear, we may denote $\mathbb{V}$ instead of $(\mathbb{V}, v)$.

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When the context is clear, we may denote $\mathbb{V}$ instead of $(\mathbb{V}, v)$.

In practice, one builds persistence modules from *filtrations.*

A family of subsets $\mathbb{X} = (X^t)_{t\in\mathbb{R}^+}$ of E is a *filtration* if it is non-decreasing for the inclusion, i.e. for any $s, t \in \mathbb{R}^+$, if $s \leq t$ then $X^s \subseteq X^t$.

In this course, we will consider filtrations of simplicial complexes, that is, non-decreasing families of simplicial complexes $\mathbb{S} = (S^t)_{t\in\mathbb{R}^+}$.

By applying the i^{th} homology functor to a filtration, we obtain a persistence module $\mathbb{V}[\mathbb{S}] = (H_i(S^t))_{t\in\mathbb{R}^+}$, with maps $((i_s^t)_* \colon H_i(S^s) \to H_i(S^t))_{s\leq t}$ induced by the inclusions.

$$\cdots \dashrightarrow S^{t_1} \xhookrightarrow{i_{t_1}^{t_2}} S^{t_2} \xhookrightarrow{i_{t_2}^{t_3}} S^{t_3} \xhookrightarrow{i_{t_3}^{t_4}} S^{t_4} \hookleftarrow \cdots$$

$$\cdots \dashrightarrow H_i(S^{t_1}) \xrightarrow{(i_{t_1}^{t_2})_*} H_i(S^{t_2}) \xrightarrow{(i_{t_2}^{t_3})_*} H_i(S^{t_3}) \xrightarrow{(i_{t_3}^{t_4})_*} H_i(S^{t_4}) \cdots$$

I - Functoriality of homology

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Isomorphisms of persistence modules

Definition: An *isomorphism* between two persistence modules $\mathbb{V}$ and $\mathbb{W}$ is a family of isomorphisms of vector spaces $\phi = (\phi_t \colon \mathbb{V}^t \to \mathbb{W}^t)_{t \in \mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$
\begin{array}{ccc}
V^s & \xrightarrow{v_s^t} & V^t \\
\downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_t} \\
W^s & \xrightarrow{w_s^t} & W^t
\end{array}
$$

Decomposability

Definition: Let $(\mathbb{V}, v)$ and $(\mathbb{W}, w)$ be two persistence modules.

Their *sum* is the persistence module $\mathbb{V} \oplus \mathbb{W}$ defined with the vector spaces $(V \oplus W)^t = V^t \oplus W^t$ and the linear maps

$$(v \oplus w)_s^t \colon (x, y) \in (V \oplus W)^s \longmapsto (v_s^t(x), w_s^t(y)) \in (V \oplus W)^t.$$

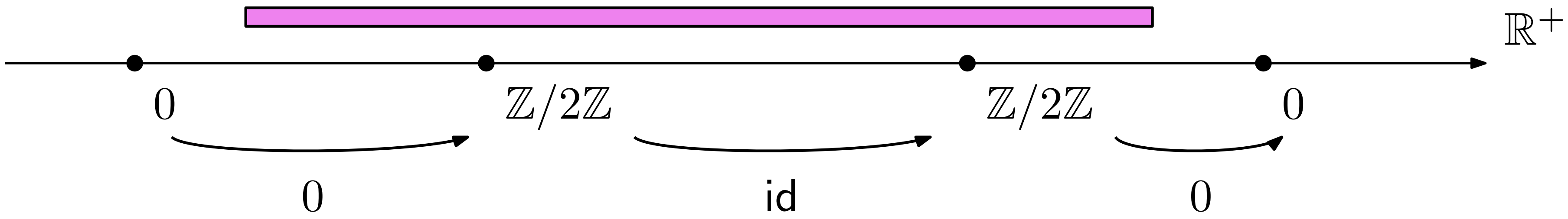
A persistence module $\mathbb{U}$ is *indecomposable* if for every pair of persistence modules $\mathbb{V}$ and $\mathbb{W}$ such that $\mathbb{U}$ is isomorphic to the sum $\mathbb{V} \oplus \mathbb{W}$, then one of the summands has to be a trivial persistence module, that is, equal to zero for every $t \in \mathbb{R}^+$.
Otherwise, $\mathbb{U}$ is said *decomposable*.

Interval modules

Definition: Let $I \subset \mathbb{R}^+$ be an interval: $[a,b]$, $(a,b]$, $[a,b)$ or (a,b), with $a,b \in \mathbb{R}^+$ such that $a \leq b$, and potentially $a = -\infty$ or $b = +\infty$.

The *interval module* associated to I is the persistence module $\mathbb{B}[I]$ with vector spaces $\mathbb{B}^t[I]$ and linear maps $v_s^t\colon \mathbb{B}^s[I] \to \mathbb{B}^t[I]$ defined as

$$\mathbb{B}^t[I] = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } t \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad v_s^t = \begin{cases} \mathrm{id} & \text{if } s,t \in I, \\ 0 & \text{otherwise.} \end{cases}$$



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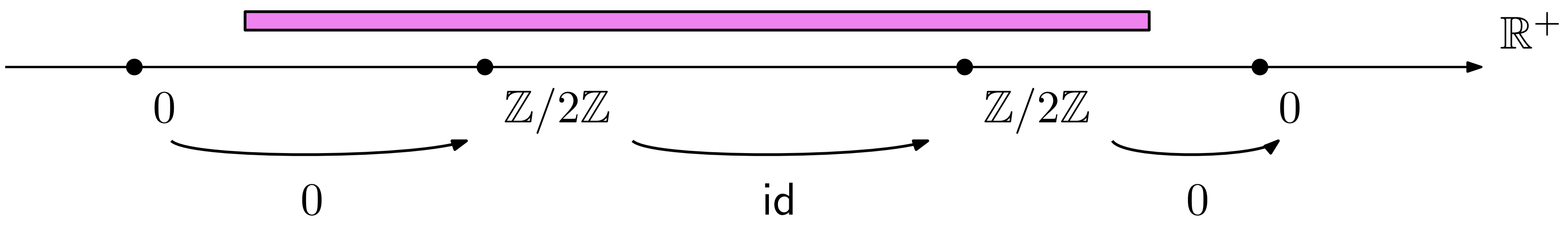

Lemma: Interval modules are indecomposable.

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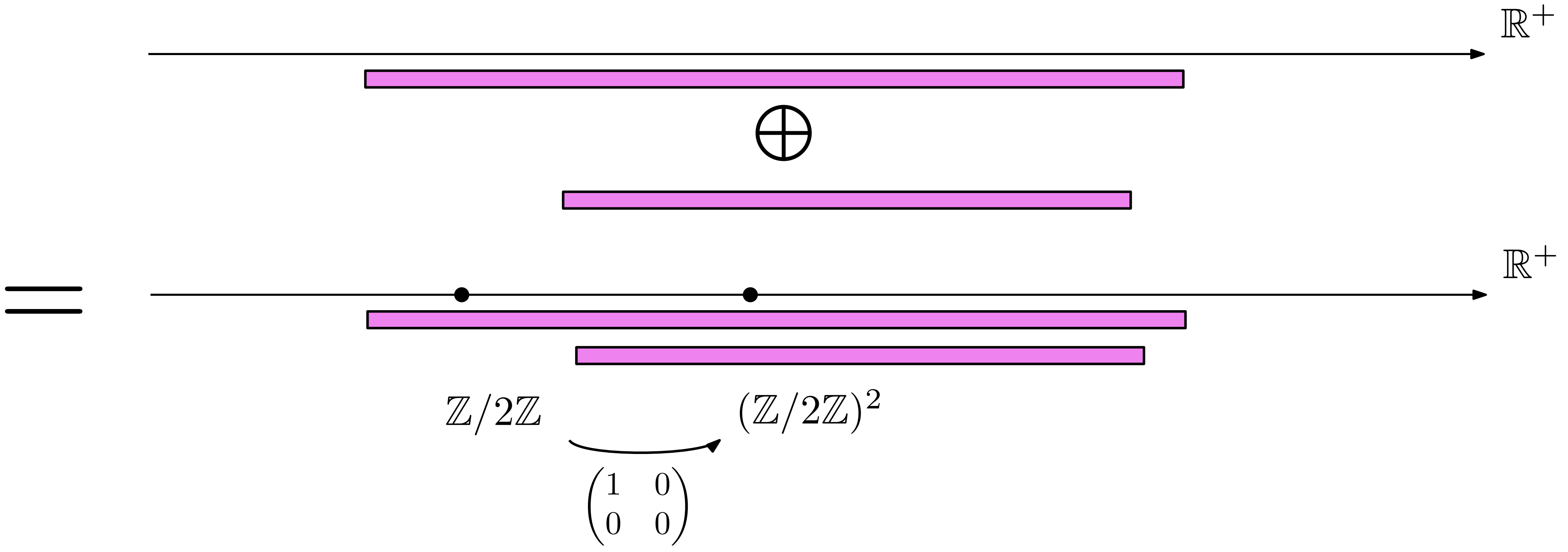
We can sum interval modules:



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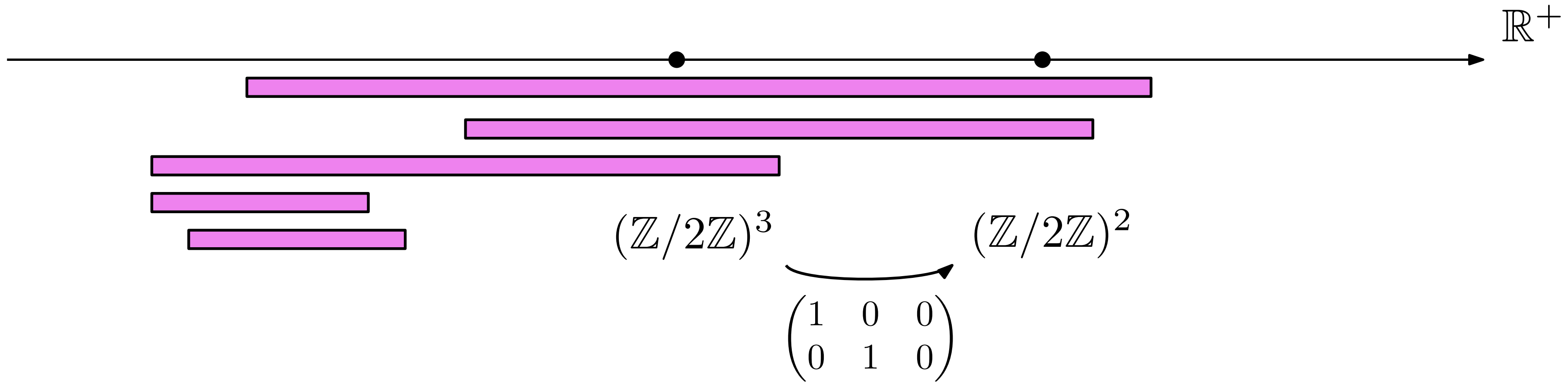


Barcodes

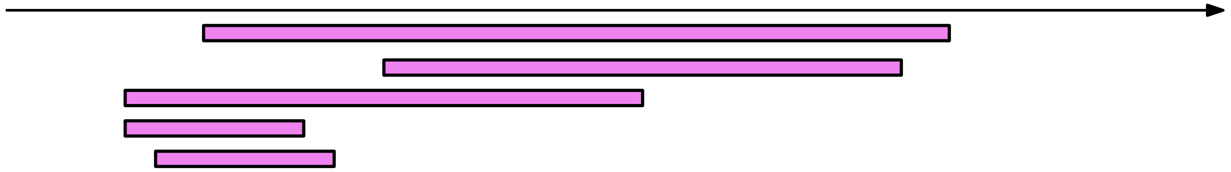
A persistence module $\mathbb{V}$ *decomposes into interval module* if there exists a multiset $\mathcal{I}$ of intervals of T such that

$$\mathbb{V} \simeq \bigoplus_{I \in \mathcal{I}} \mathbb{B}[I].$$

Multiset means that $\mathcal{I}$ may contain several copies of the same interval I.

Theorem (consequence of Krull–Remak–Schmidt–Azumaya): If a persistence module decomposes into interval modules, then the multiset $\mathcal{I}$ of intervals is unique.

In this case, $\mathcal{I}$ is called the *persistence barcode* of $\mathbb{V}$. It is written $\mathrm{Barcode}\,(\mathbb{V})$.



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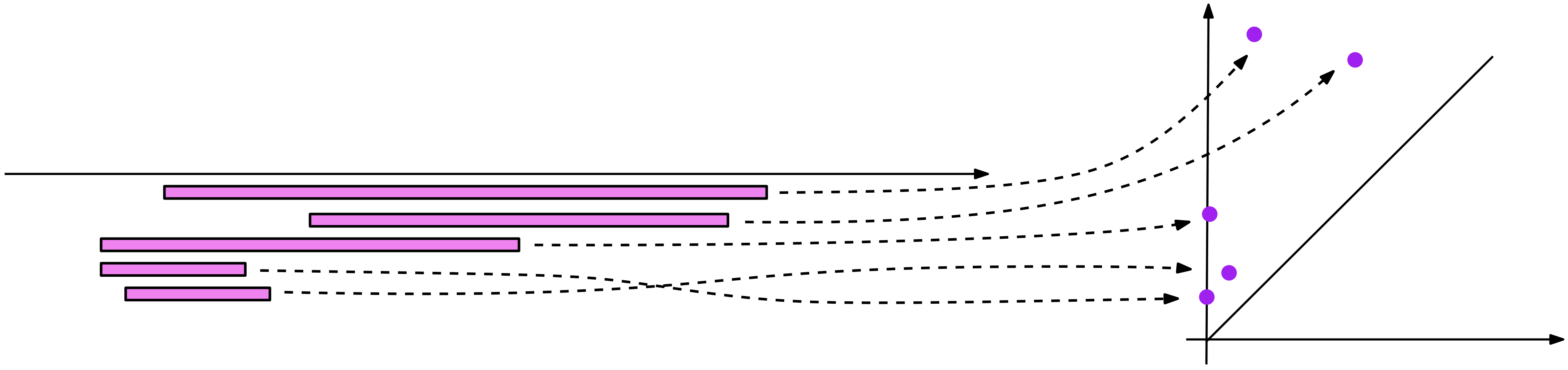
For every $[a, b]$, $(a, b]$, $[a, b)$ or (a, b) in $\mathrm{Barcode}\,(\mathbb{V})$, consider the point (a, b) of $\mathbb{R}^2$. The collection of all such points is the *persistence diagram* of $\mathbb{V}$.



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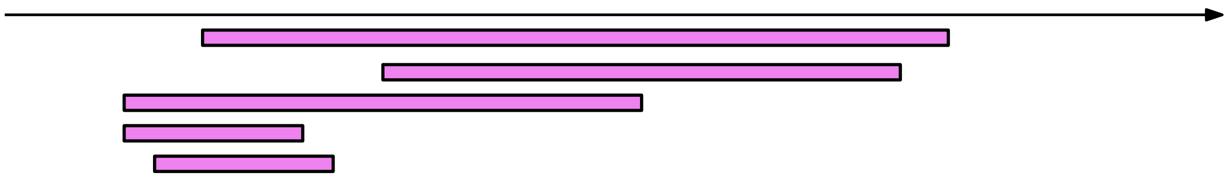
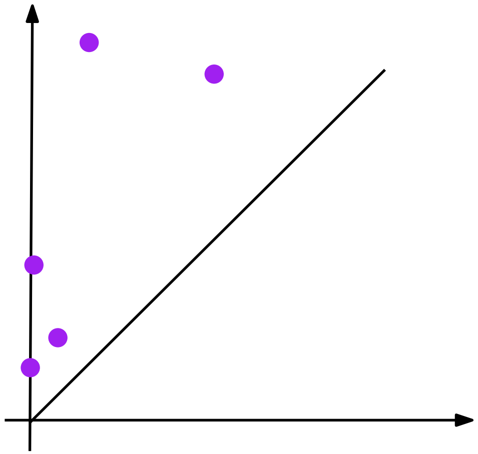


$\mathrm{Barcode}(\mathbb{V})$



$\mathrm{Diagram}(\mathbb{V})$

Barcodes

A persistence module $\mathbb{V}$ is said *pointwise finite dimensional* if $\dim V^t < +\infty$ for all t.

Theorem (Crawley-Boevey, 2015): Every pointwise finite-dimensional persistence module decomposes into interval modules.

Barcodes

A persistence module $\mathbb{V}$ is said *pointwise finite dimensional* if $\dim V^t < +\infty$ for all t.

Theorem (Crawley-Boevey, 2015): Every pointwise finite-dimensional persistence module decomposes into interval modules.

Proof (Zomorodian, Carlsson, 2005): **Simpler case:** the persistence module is finite-dimensional *and* has finitely many terms.
We can write our persistence module as

$$V^1 \xrightarrow{v_1^2} V^2 \xrightarrow{v_2^3} V^3 \xrightarrow{v_3^4} V^4 \dashrightarrow \dots \dashrightarrow V^n$$

Consider the vector space $\mathcal{V} = \bigotimes_{1 \le i \le n} V^i = V^1 \times \cdots \times V^n$.
Let $\mathbb{Z}/2\mathbb{Z}[x]$ denote the space of polynomials with coefficients in $\mathbb{Z}/2\mathbb{Z}$. We give $\mathcal{V}$ an action of $\mathbb{Z}/2\mathbb{Z}[x]$ via

$$x \cdot (a^1, a^2, ..., a^n) = (0, v_1^2(a^1), v_2^3(a^2), ..., v_{n-1}^n(a^{n-1})).$$

Hence $\mathcal{V}$ can be seen as a finitely generated module over the principal ideal domain $\mathbb{Z}/2\mathbb{Z}[x]$. By classification, $\mathcal{V}$ is isomorphic to a sum

$$\mathcal{V} \simeq \bigoplus_{i \in I} \mathbb{Z}/2\mathbb{Z}[x]/x^i \cdot \mathbb{Z}/2\mathbb{Z}[x].$$

We identify the components $\mathbb{Z}/2\mathbb{Z}[x]/x^i \cdot \mathbb{Z}/2\mathbb{Z}[x]$ with bars of the barcode of length i.

Barcodes

Barcodes

On a barcode we can read homology **at each step**, and see how it **evolves**.

The Čech or the Rips filtration define an increasing sequence of simplices

$$\ldots \subset \check{\mathrm{C}}\mathrm{ech}^{t_1}(X) \subset \check{\mathrm{C}}\mathrm{ech}^{t_2}(X) \subset \check{\mathrm{C}}\mathrm{ech}^{t_3}(X) \subset \ldots$$

We can turn it consistently into an ordering of the simplices, by inserting the simplices by order of apparition in the filtration.

$$\sigma^1 < \sigma^2 < \ldots < \sigma^n$$

Denote $t(\sigma)$ the time of apparition of the simplex σ in the filtration. The total order on the simplices satisfies

$$t(\sigma^i) < t(\sigma^j) \text{ for all } i < j.$$

In practice several simplices may appear at the same time. If this occurs, choose an order of the simplices.

⟶ Consider the boundary matrix, and compute a Gauss reduction.

Algorithm

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	σ^8	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	1	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^5+\sigma^6+\sigma^7+\sigma^8$	$\sigma^9+\sigma^6+\sigma^7$	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Algorithm

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	σ^8	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	1	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

For any $j \in [\![1, n]\!]$,

$$\delta(j) = \max\{i \in [\![1, n]\!], \Delta_{i,j} \neq 0\},$$

and $\Delta_{i,j} = 0$ for all j, then $\delta(j)$ is *undefined*.

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^5 + \sigma^6 + \sigma^7 + \sigma^8$	$\sigma^9 + \sigma^6 + \sigma^7$	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Algorithm

Now, for all j such that $\delta(j)$ is defined, consider the pair of simplices

$$\left(\sigma^{\delta(j)}, \sigma^{j}\right).$$

Also, for all i such that $\forall j, \delta(j) \neq i$, we set: $\left(\sigma^{i}, +\infty\right)$.

The pairs of simplices (σ, τ) are called *persistence pairs*.

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^5+\sigma^6+\sigma^7+\sigma^8$	$\sigma^9+\sigma^6+\sigma^7$	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
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σ^1 ($+\infty$)	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
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σ^5	0	0	0	0	0	0	0	0	0	1
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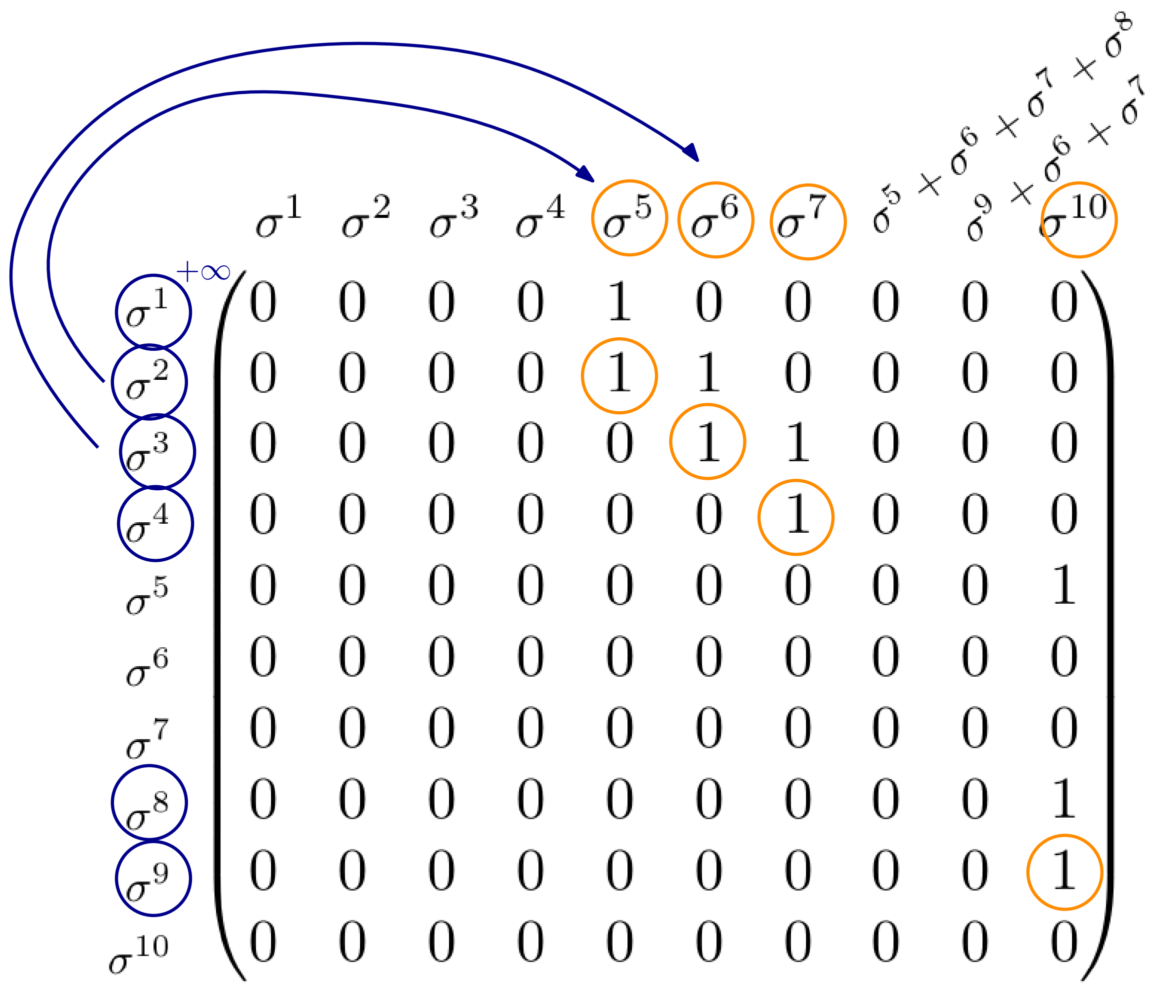

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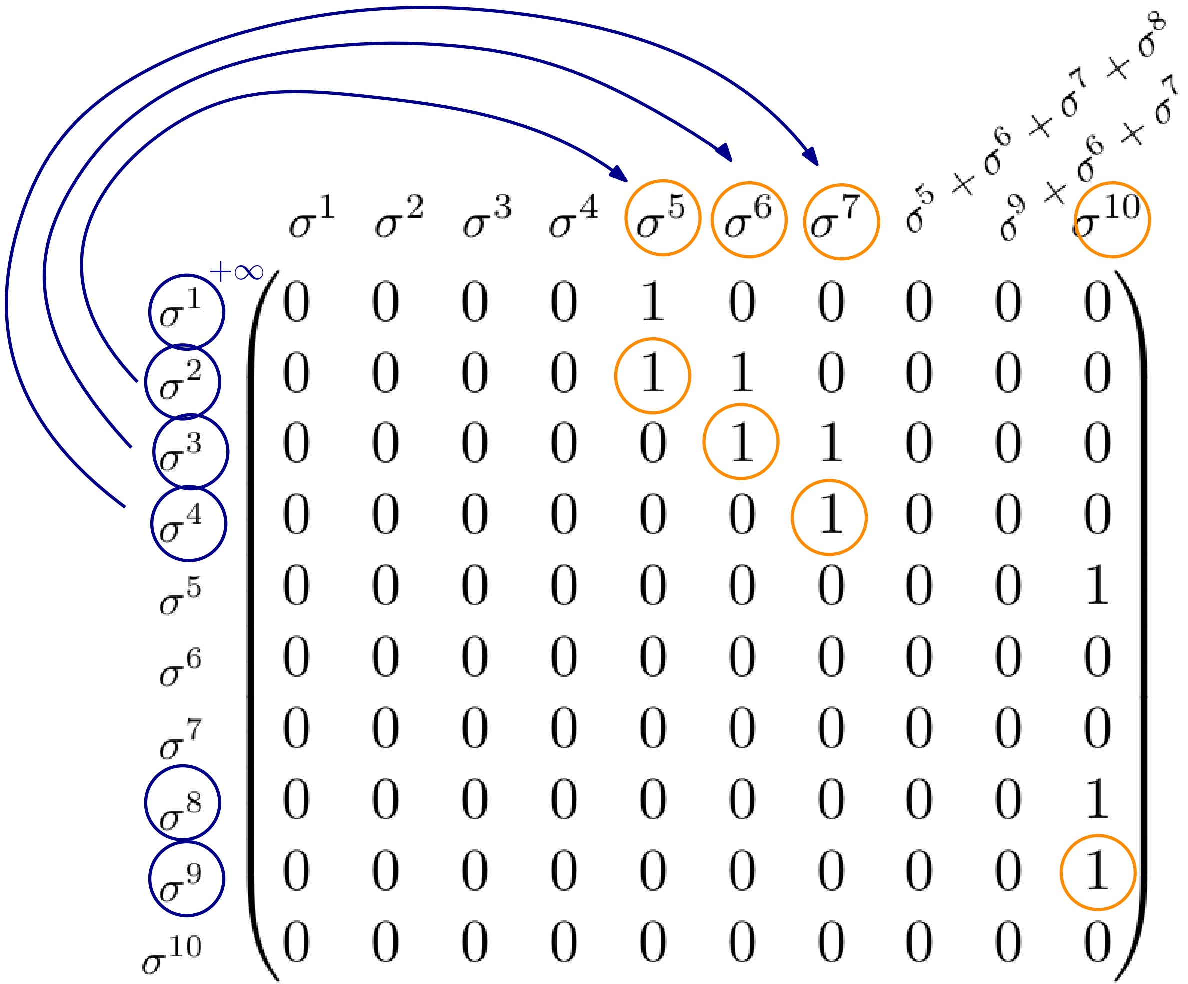

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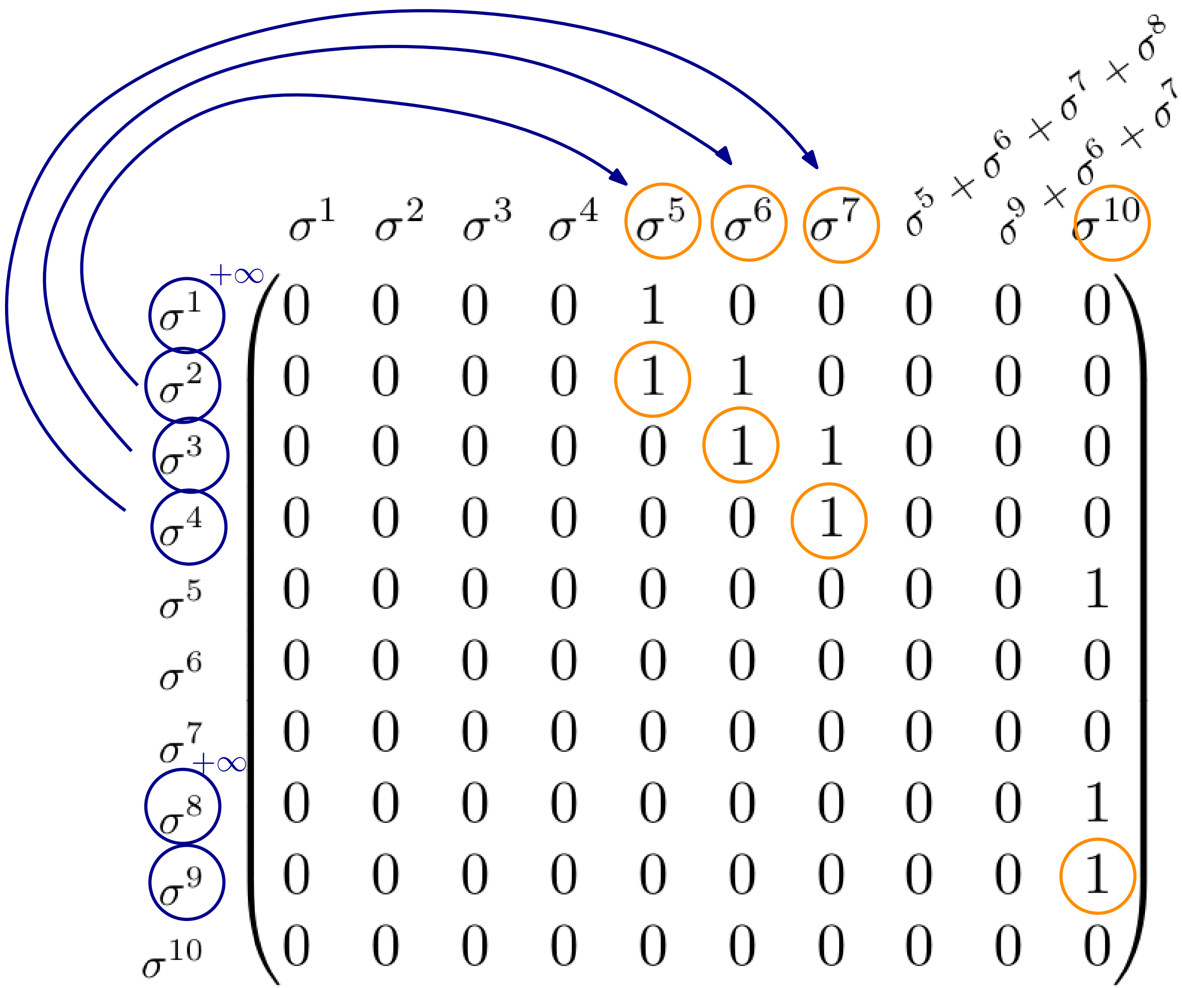

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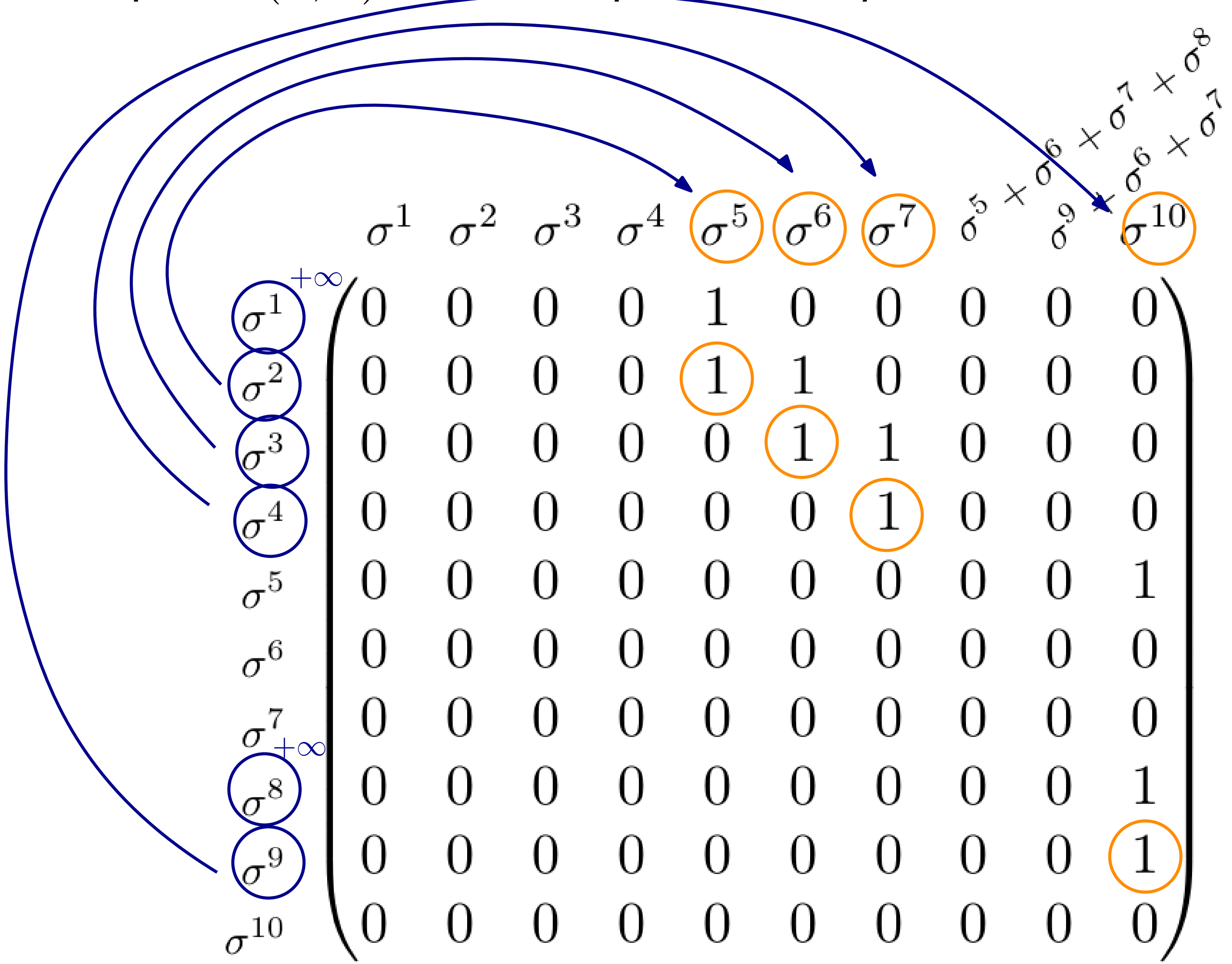

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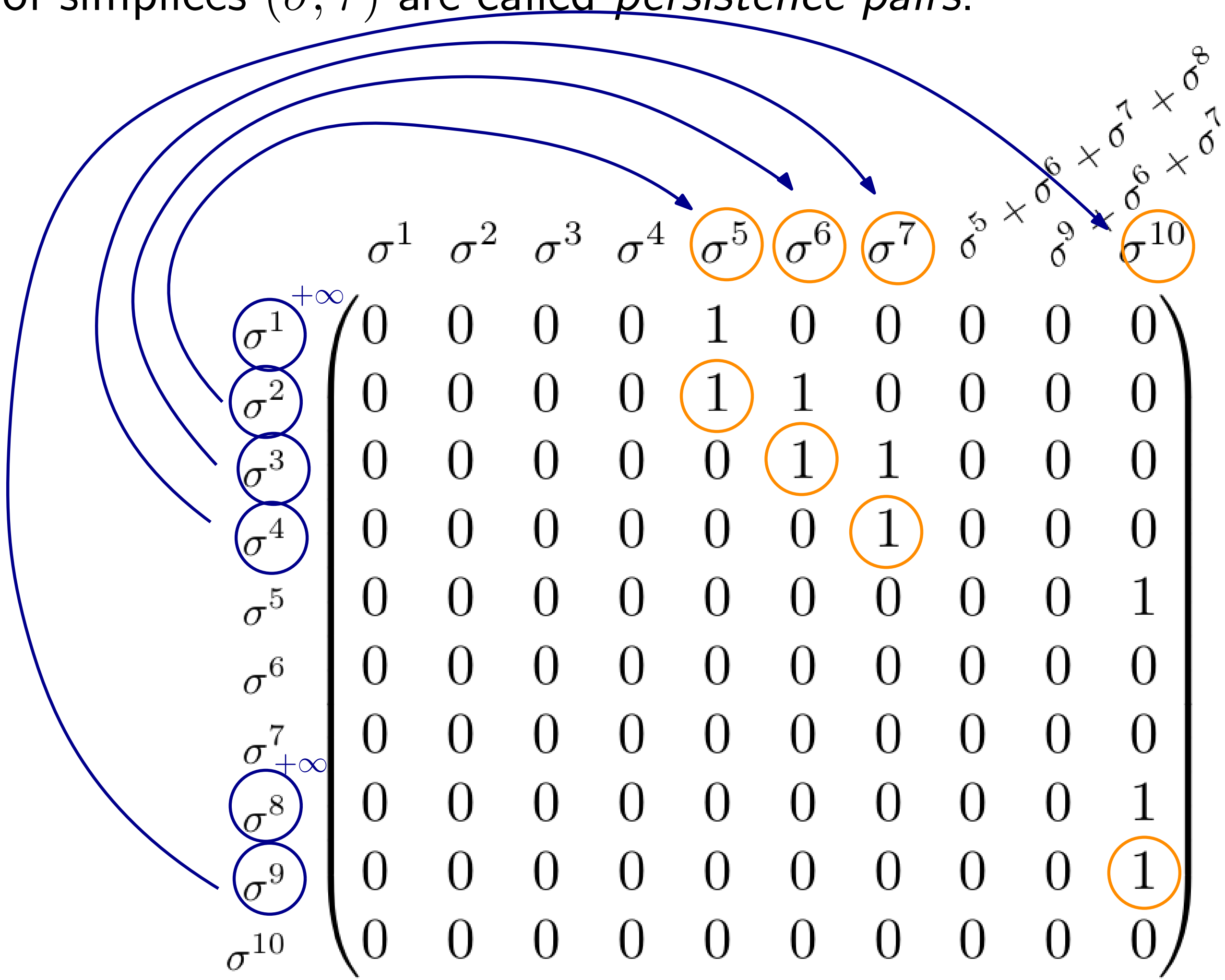

$(\sigma^1, +\infty)$

(σ^2, σ^5)

(σ^3, σ^6)

(σ^4, σ^7)

$(\sigma^8, +\infty)$

$(\sigma^9, \sigma^1 0)$

Algorithm

Proposition: The barcodes of the filtration consists in the intervals

$$\mathcal{I} = \left\{ (t(\sigma), t(\tau)) \ \text{ for all persistence pair } \ (\sigma, \tau) \ \text{ such that } \ t(\sigma) \neq t(\tau) \right\}.$$

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^5+\sigma^6+\sigma^7+\sigma^8$	$\sigma^9+\sigma^6+\sigma^7$	σ^{10}
σ^1 $+\infty$	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
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Algorithm

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Proof: We shall show that the algorithm allows to define, for all $i, j \geq 0$, a basis $\mathcal{B}_i^j$ of $H_i(K^j)$, such that one passes from $\mathcal{B}_i^j$ to $\mathcal{B}_i^{j+1}$ by adding or removing a cycle.

As a consequence, we obtain an isomorphism between the persistence module and a sum of interval modules given by $\mathcal{I}$.

We build the basis as follows: for every $j \geq 0$, consider the simplex σ^j and its dimension $i = \dim(\sigma^j)$.

If σ^j is positive, then we add the corresponding cycle to the basis $\mathcal{B}_i^{j-1}$.

If it is negative, then there exists a simplex σ^k, with $k < j$, such that $\delta(k) = j$. We remove the cycle corresponding to σ^k to the basis $\mathcal{B}_{i-1}^{j-1}$.

Conclusion

We used induced maps in homology to track the cycles throughout filtrations.

We gathered all this information into a persisten ce module.

We have seen that the barcode of a persistence module summarizes the persistence of all the cycles.

We used the incremental algorithm to compute the barcode.

Homework: Exercise 52
Facultative: Exercises 48, 49, 51

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Merci !