EMAp Summer Course

Topological Data Analysis with Persistent Homology

https://raphaeltinarrage.github.io/EMAp.html

Lesson 6: Incremental algorithm

Last update: February 2, 2021

Introduction

Yesterday we have defined

chain complex	$\dots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots$
n-cycles	$Z_n(K) = \mathsf{Ker}(\partial_n)$
n-boundaries	$B_n(K) = \mathsf{Im}(\partial_{n+1})$
n^{th} homology group	$H_n(K) = Z_n(K)/B_n(K)$

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n-boundaries $B_n(K) = \mathsf{Im}(\partial_{n+1})$

n^{th} homology group $\boxed{H_n(K) = Z_n(K)/B_n(K)}$

Today's objectives: how to compute them? what do they represent?

I - Incremental algorithm

II - Applications

III - Matrix algorithm

Ordering the simplicial complex

Let K be a simplicial complex with n simplices. Choose a total order of the simplices

$$\sigma^1 < \sigma^2 < ... < \sigma^n$$

such that

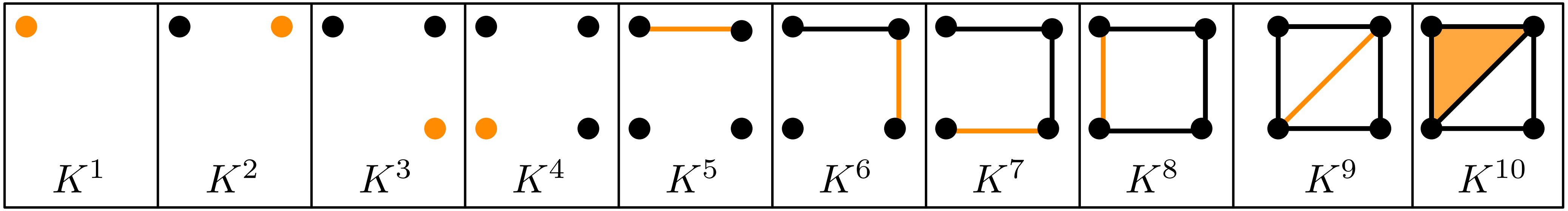
$$\forall \sigma, \tau \in K,\ \tau \subsetneq \sigma \implies \tau < \sigma.$$

In other words, a face of a simplex is lower than the simplex itself.
For every $i \leq n$, consider the simplicial complex

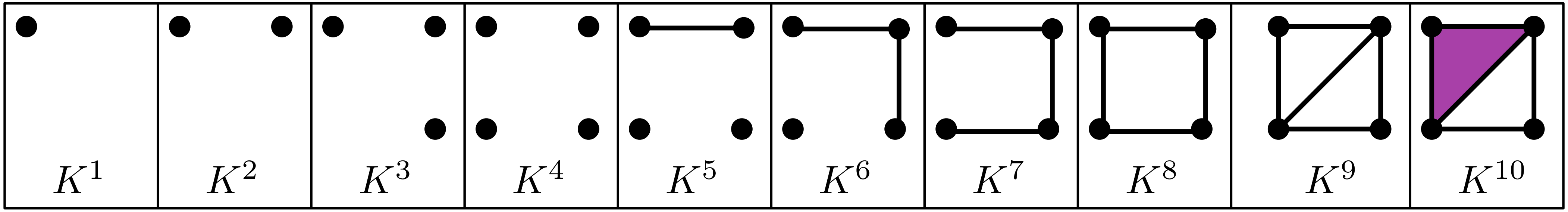
$$K^i = \{\sigma^1, ..., \sigma^i\}.$$

We have $\forall i \leq n, K^{i+1} = K^i \cup \left\{\sigma^{i+1}\right\}$, and $K^n = K$. They form an inscreasing sequence of simplicial complexes

$$K^1 \ \subset \ K^2 \ \subset \ ... \ \subset \ K^n.$$

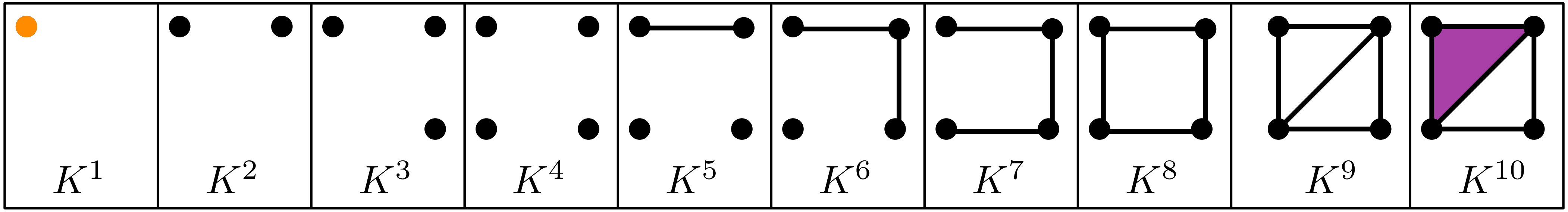


Positivity of simplices



Let $k \geq 0$. We will compute the homology groups of K^i incrementally:

$$H_k(K^1),\ H_k(K^2),\ H_k(K^3),\ H_k(K^4),\ H_k(K^5),\ H_k(K^6),\ H_k(K^7),\ H_k(K^8),\ H_k(K^9),\ H_k(K^{10})$$



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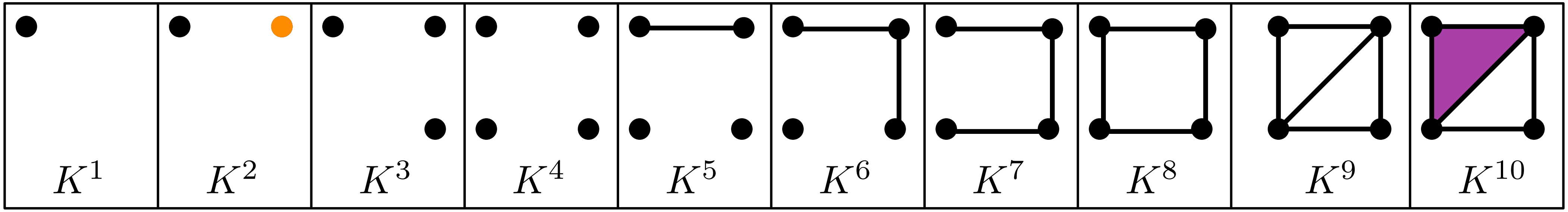
Definition: Let $i \in [\![1, n]\!]$, and $d = \dim(\sigma^i)$. Recall that $K^i = K^{i-1} \cup \{\sigma_i\}$.
The simplex σ^i is *positive* if there exists a cycle $c \in Z_d(K^i)$ that contains σ^i.
In other words, there exist $c = \sum_{\sigma \in K^i_{(n)}} \epsilon_\sigma \cdot \sigma \in C_n(K^i)$ such that $\epsilon_{\sigma^i} = 1$ and $\partial_n(c) = 0$. Otherwise, σ^i is *negative*.

Example:

- $\sigma^1 \in K^1$ is **positive** because it is included in the cycle $c = \sigma^1$ (indeed, $\partial_0(\sigma^1) = 0$).

Positivity of simplices



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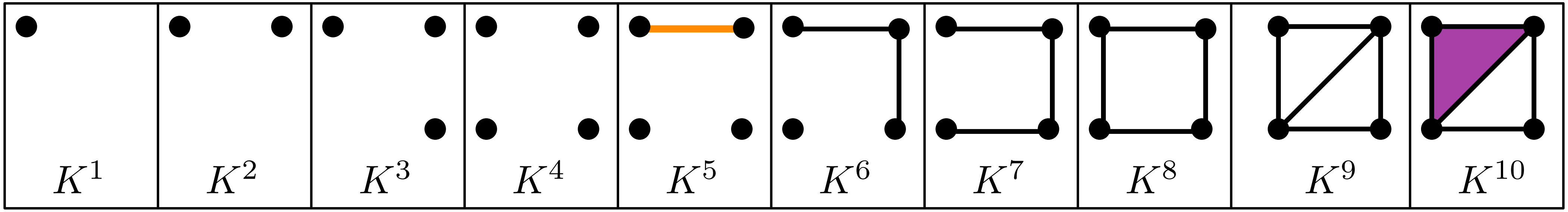
Positivity of simplices



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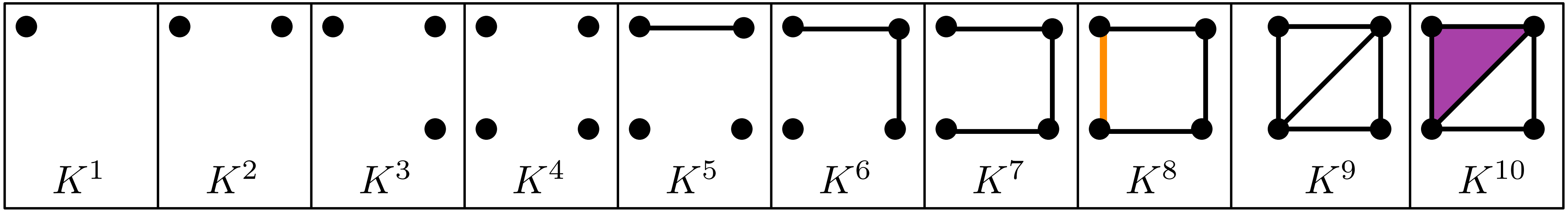
Positivity of simplices



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- $\sigma^8 \in K^8$ is **positive** because it is included in the cycle $c = \sigma^5 + \sigma^6 + \sigma^7 + \sigma^8$ (indeed, $\partial_1(c) = 2\sigma^1 + 2\sigma^2 + 2\sigma^3 + 2\sigma^4 = 0$).

Positivity of simplices

Definition: Let $i \in [\![1, n]\!]$, and $d = \dim(\sigma_i)$. Recall that $K^i = K^{i-1} \cup \{\sigma_i\}$.
The simplex σ_i is *positive* if there exists a cycle $c \in Z_d(K^i)$ that contains σ_i.
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Remark: By adding σ^i in the simplicial complex, the only groups that may change are $Z_d(K^i)$ and $B_{d-1}(K^i)$.

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Lemma: If σ^i is positive, then $\beta_d(K^i) = \beta_d(K^{i-1}) + 1$,
and for all $d' \neq d$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.

Proof: We start by proving the following fact: if $c \in Z_d(K^i)$ is a cycle that contains σ_i, then c is not homologous (in K^i) to a cycle of $c' \in Z_d(K^{i-1})$.

By contradiction: if $c = c' + b$ with $c' \in Z_d(K^{i-1})$ and $b \in B_d(K^i)$, then $c - c' = b \in B_d(K^i)$. This is absurd because we just added σ_i: it cannot appear in a boundary of K^i.

As a consequence, $\dim Z_d(K^i) = \dim Z_d(K^{i-1}) + 1$.

We conclude by using the relation $\beta_d(K^i) = \dim Z_d(K^i) - \dim B_d(K^i)$.

Positivity of simplices

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Remark: By adding σ^i in the simplicial complex, the only groups that may change are $Z_d(K^i)$ and $B_{d-1}(K^i)$.

Lemma: If σ^i is negative, then $\beta_{d-1}(K^i) = \beta_{d-1}(K^{i-1}) - 1$,
and for all $d' \neq d-1$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.

Proof: We start by proving the following fact: $\partial_d(\sigma^i)$ is not a boundary of K^{i-1}.

Otherwise, we would have $\partial_d(\sigma^i) = \partial_d(c)$ with $c \in C_d(K^{i-1})$, i.e. $\partial_d(\sigma^i + c) = 0$.
Hence $\sigma^i + c$ would be a cycle of K^i that contains c, contradicting the negativity of σ^i.

As a consequence, $\dim B_{d-1}(K^i) = \dim B_{d-1}(K^{i-1}) + 1$.

We conclude by using the relation $\beta_{d-1}(K^i) = \dim Z_{d-1}(K^i) - \dim B_{d-1}(K^i)$.

Incremental algorithm

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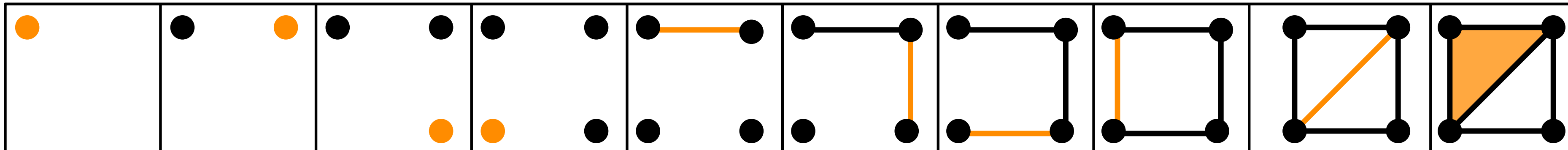
We deduce the following algorithm:

Input: an increasing sequence of simplicial complexes $K^1 \subset \cdots \subset K^n = K$
Output: the Betti numbers $\beta_0(K), ...\beta_d(K)$
$\beta_0 \leftarrow 0, ..., \beta_d \leftarrow 0$;
for $i \leftarrow 1$ **to** n **do**
  $d = \dim(\sigma^i)$;
  if σ^i *is positive* **then**
    $\beta_k(K^i) \leftarrow \beta_k(K^i) + 1$;
  else if $d > 0$ **then**
    $\beta_{k-1}(K^i) \leftarrow \beta_{k-1}(K^{i-1}) - 1$;

Incremental algorithm



	K^1	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}
Dimension	0	0	0	0	1	1	1	1	1	2
Positivity	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$
$\beta_0(K^i)$	1	2	3	4	3	2	1	1	1	1
$\beta_1(K^i)$	0	0	0	0	0	0	0	1	2	1

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I - Incremental algorithm

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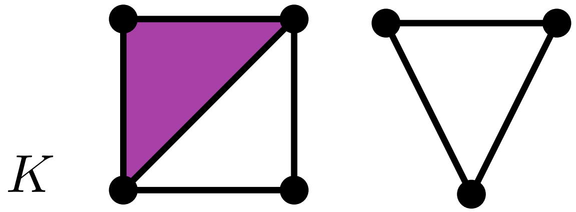
Number of connected components

Proposition: Let X be a (triangulable) topological space. Then its 0^{th} Betti number, $\beta_0(X)$, is equal to the number of connected components of X.

Let K be a triangulation of X.



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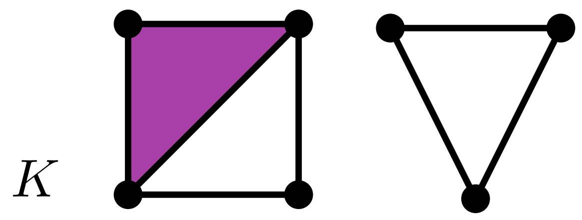

First, a definition: say that a simplicial complex L is *combinatorially connected* of for every vertex v, w of L, there exists a sequence of edges that connects v and w:

$$[v, v_1], \quad [v_1, v_2], \quad [v_2, v_3], \quad ..., \quad [v_n, w].$$

Let m be the number of connected components X, and let K be triangulation of X. We accept the following equivalent statement: there exists m **disjoint**, **non-empty** and **combinatorially connected** simplicial sub-complex $L_1, ..., L_m$ of K such that

$$K = \bigcup_{1 \leq i \leq m} L_i.$$

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Proposition: Let X be a (triangulable) topological space. Then its 0^{th} Betti number, $\beta_0(X)$, is equal to the number of connected components of X.

Let K be a triangulation of X.

Proof: Let T be a spanning forest of K, that is, a union of spanning trees. It admits m combinatorially connected components.

Consider an ordering of the simplices of K that begins with an ordering of T.

Apply the incremental algorithm. Each vertex increases β_0 by 1.

Since T is a tree, all its edges are negative simplices (T has no cycles), hence decrease β_0. Each tree of the forest contains $k-1$ edges, where k is the number of vertices of the corresponding component.

Since T is a spanning tree, each other edges of K is positive, hence β_0 does not change. Similarly, the other simplices of K do not change β_0. We deduce the result.

Homology of spheres

For any $n \geq 1$, consider the vertex set $V = \{0, \ldots, n\}$, and the simplicial complex

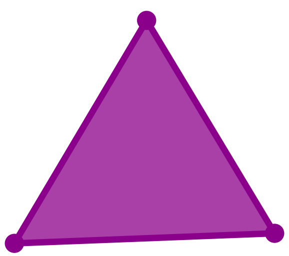
$$\Delta_n = \{S \subset V, S \neq \emptyset\}.$$

We call it the *simplicial standard n-simplex*. Define its boundary as

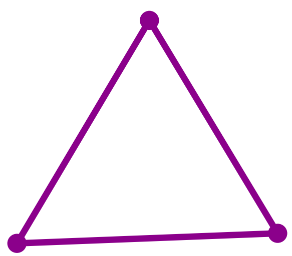
$$\partial\Delta_n = \Delta_n \setminus V.$$

The simplicial complex $\partial\Delta_n$ is a triangulation of the $(n-1)$-sphere $\mathbb{S}_{n-1} \subset \mathbb{R}^n$.

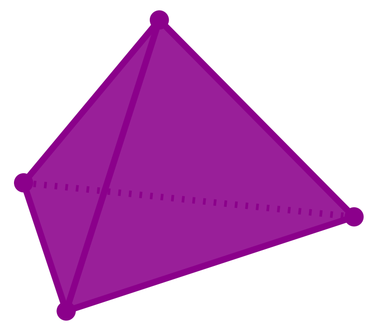
As a consequence, for all $i \geq 0$, $H_i(\mathbb{S}_n) = H_i(\partial\Delta_{n+1})$.



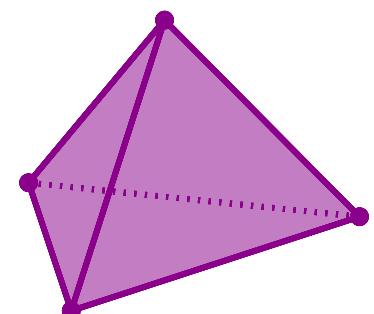
Δ_2



$\partial\Delta_2$



Δ_3



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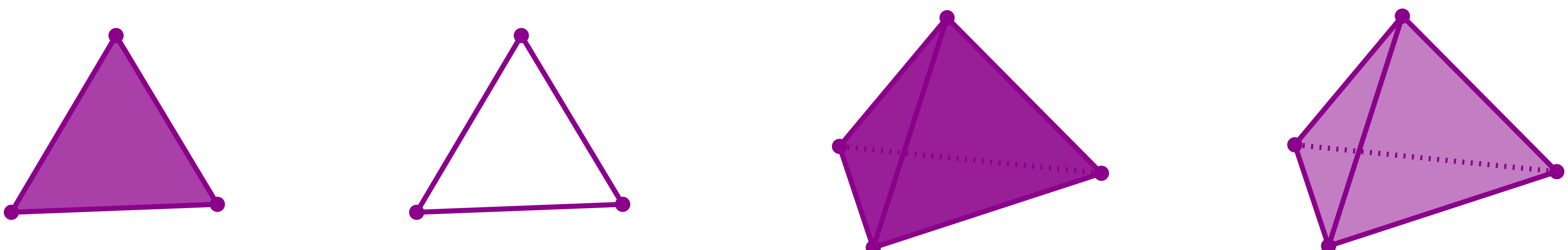


Δ_2 $\partial\Delta_2$ Δ_3 $\partial\Delta_3$

Proposition: The Betti numbers of $\mathbb{S}_n$ are:

- $\beta_i(\mathbb{S}_n) = 1$ for $i = 0$ or n,
- $\beta_i(\mathbb{S}_n) = 0$ else.

Homology of spheres

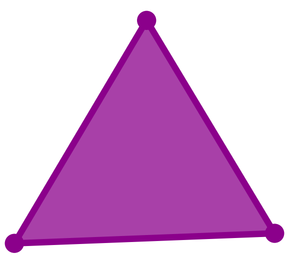
Proof: Consider the *simplicial standard n-simplex* Δ_n. It is homotopy equivalent to a point (its topological realization deform retracts on any point of it). Hence Δ_n has the same Betti numbers as the point:

- $\beta_1(\Delta_n) = 1$,
- $\beta_i(\Delta_n) = 0$ for $i > 0$.

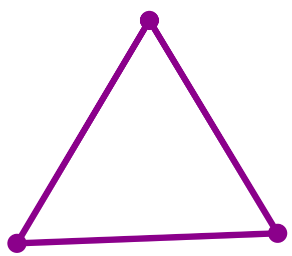
Now, if we run the incremental algorithm for homology on Δ_n, but stopping before adding the n-simplex V, we would obtain the Betti numbers of $\partial\Delta_n$.

Note that the n-simplex is negative. Hence

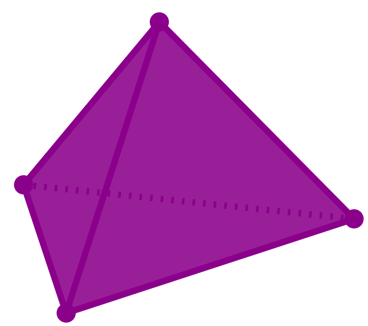
- $\beta_n(\partial\Delta_n) = \beta_n(\Delta_n) + 1$,
- $\beta_i(\partial\Delta_n) = \beta_i(\Delta_n)$ for $i \neq n$.



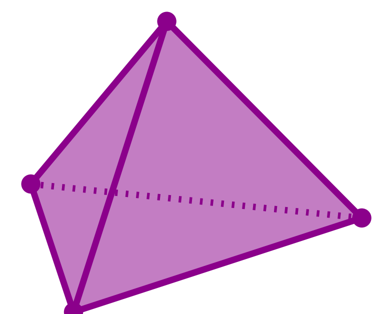
Δ_2



$\partial\Delta_2$



Δ_3



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Invariance of domain

Theorem (Invariance of Domain): For every integers m, n such that $m \neq n$, the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

Proof: Let m, n such that $m \neq n$. By contradiction, suppose that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic via f.

Let 0 denote the origin of $\mathbb{R}^n$. By restriction, we get a homeomorphism

$$\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}.$$

We deduce the following weaker statement: $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{f(0)\}$ are homotopic equivalent.

We deduce that the sphere $\mathbb{S}_{n-1}$ and $\mathbb{S}_{m-1}$ are homotopic equivalent.

Hence $\mathbb{S}_{n-1}$ and $\mathbb{S}_{m-1}$ must admit the same homology groups. This contradict the previous proposition.

Euler characteristic

Reminder: the Euler characteristic of a simplicial complex K is

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Proposition: Let X be a (triangulable) topological space. Then its Euler characteristic is equal to

$$\chi(X) = \sum_{0 \leq i \leq n} (-1)^i \cdot \beta_i(X)$$

where n is the maximal integer such that $\beta_i(X) \neq 0$.

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Proof: Let K be a triangulation of X. Pick an ordering $K^1 \subset \cdots \subset K^n = K$ of K, with $K^i = K^{i-1} \cup \{\sigma^i\}$ for all $2 \leq i \leq n$.

By induction, let us show that, for all $1 \leq m \leq n$,

$$\sum_{0 \leq i \leq m} (-1)^i \cdot \beta_i(K^m) = \sum_{0 \leq i \leq m} (-1)^i \cdot (\text{number of simplices of dimension } i \text{ of } K^m).$$

For $m = 1$, σ^m is a 0-simplex, and the equality reads $1 = 1$.

Now, suppose that the equality is true for $1 \leq m < n$, and consider the simplex σ^{m+1}. Let $d = \dim \sigma^{m+1}$. The right-hand side of the Equation is increased by $(-1)^d$.

If σ^{m+1} is positive, then $\beta_d(K^{m+1}) = \beta_d(K^m) + 1$, hence the left-hand side of the Equation is increased by $(-1)^d$.

Otherwise, it is negative, and $\beta_{d-1}(K^{m+1}) = \beta_{d-1}(K^m) - 1$, hence the left-hand side of the Equation is increased by $-(-1)^{d-1} = (-1)^d$.

I - Incremental algorithm

II - Applications

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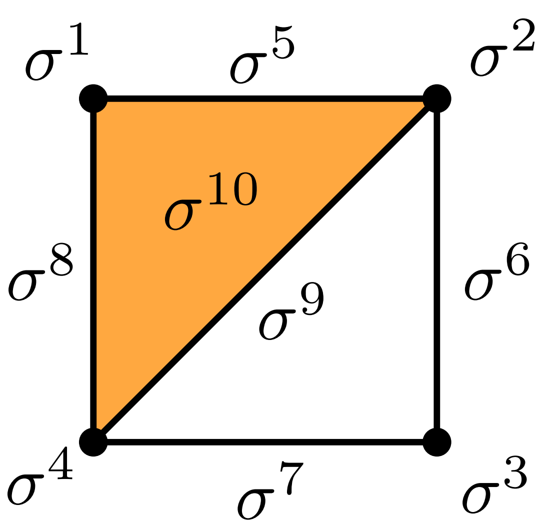
Boundary matrix

The only thing missing to apply the incremental algorithm is to determine whether a simplex is positive or negative.

Let K be a simplicial complex, and $\sigma^1 < \sigma^2 < \cdots < \sigma^n$ and ordering of its simplices.

Define the *boundary matrix* of K, denoted Δ, as follows: Δ is a $n \times n$ matrix, whose (i, j)-entry (i^{th} row, j^{th} column is)

$$\Delta_{i,j} = \begin{cases} 1 & \text{if } \sigma^i \text{ is a face of } \sigma^j \text{ and } |\sigma^i| = |\sigma^j| - 1 \\ 0 & \text{else.} \end{cases}$$



$$\begin{array}{c} \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Boundary matrix

By adding columns one to the others, we create chains.
If we were able to reduce a column to zero, then we found a cycle.

$$\begin{array}{c|cccccccccc} & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \\ \hline \sigma^1 & 0&0&0&0&1&0&0&1&0&0 \\ \sigma^2 & 0&0&0&0&1&1&0&0&1&0 \\ \sigma^3 & 0&0&0&0&0&1&1&0&0&0 \\ \sigma^4 & 0&0&0&0&0&0&1&1&1&0 \\ \sigma^5 & 0&0&0&0&0&0&0&0&0&1 \\ \sigma^6 & 0&0&0&0&0&0&0&0&0&0 \\ \sigma^7 & 0&0&0&0&0&0&0&0&0&0 \\ \sigma^8 & 0&0&0&0&0&0&0&0&0&1 \\ \sigma^9 & 0&0&0&0&0&0&0&0&0&1 \\ \sigma^{10} & 0&0&0&0&0&0&0&0&0&0 \end{array}$$

$$\partial_1(\sigma^6) = \sigma^2 + \sigma^3$$

$$\begin{array}{c|cccccccccc} & \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9 & \sigma^{10} \\ \hline \sigma^1 & 0&0&0&0&1&0&0&0&0&0 \\ \sigma^2 & 0&0&0&0&1&1&0&0&1&0 \\ \sigma^3 & 0&0&0&0&0&1&1&0&0&0 \\ \sigma^4 & 0&0&0&0&0&0&1&0&1&0 \\ \sigma^5 & 0&0&0&0&0&0&0&0&0&1 \\ \sigma^6 & 0&0&0&0&0&0&0&0&0&0 \\ \sigma^7 & 0&0&0&0&0&0&0&0&0&0 \\ \sigma^8 & 0&0&0&0&0&0&0&0&0&1 \\ \sigma^9 & 0&0&0&0&0&0&0&0&0&1 \\ \sigma^{10} & 0&0&0&0&0&0&0&0&0&0 \end{array}$$

$$\partial_1(\sigma^5 + \sigma^6 + \sigma^7 + \sigma^8) = 0$$

Boundary matrix

The process of reducing columns to zero is called *Gauss reduction*.
For any $j \in [\![1, n]\!]$, define
$$\delta(j) = \max\{i \in [\![1, n]\!], \Delta_{i,j} \neq 0\}.$$

If $\Delta_{i,j} = 0$ for all j, then $\delta(j)$ is *undefined*.

We say that the boundary matrix Δ is *reduced* if the map δ is injective on its domain of definition.

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^5+\sigma^6+\sigma^7+\sigma^8$	$\sigma^9+\sigma^6+\sigma^7$	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Boundary matrix

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ

Output: a reduced matrix $\widetilde{\Delta}$

for $j \leftarrow 1$ **to** n **do**

  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**

    add column i to column j;

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	σ^8	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	1	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8 + \sigma^7$	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	1	0	0
σ^4	0	0	0	0	0	0	1	0	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Boundary matrix

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8+\sigma^7$	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	1	0	0
σ^4	0	0	0	0	0	0	1	0	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8+\sigma^7+\sigma^6$	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	1	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Boundary matrix

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8 + \sigma^7 + \sigma^6$	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	1	0	0
σ^2	0	0	0	0	1	1	0	1	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8 + \sigma^7 + \sigma^6 + \sigma^5$	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Boundary matrix

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8 + \sigma^7 + \sigma^6 + \sigma^5$	σ^9	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	1	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	1	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^8 + \sigma^7 + \sigma^6 + \sigma^5$	$\sigma^9 + \sigma^6 + \sigma^7$	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0

Boundary matrix

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $j \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

Lemma: Suppose that the boundary matrix is reduced. Let $j \in [\![1, n]\!]$.
If $\delta(j)$ is defined, then the simplex σ^j is negative.
Otherwise, it is positive.

	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	$\sigma^5+\sigma^6+\sigma^7+\sigma^8$	$\sigma^9+\sigma^6+\sigma^7$	σ^{10}
σ^1	0	0	0	0	1	0	0	0	0	0
σ^2	0	0	0	0	1	1	0	0	0	0
σ^3	0	0	0	0	0	1	1	0	0	0
σ^4	0	0	0	0	0	0	1	0	0	0
σ^5	0	0	0	0	0	0	0	0	0	1
σ^6	0	0	0	0	0	0	0	0	0	0
σ^7	0	0	0	0	0	0	0	0	0	0
σ^8	0	0	0	0	0	0	0	0	0	1
σ^9	0	0	0	0	0	0	0	0	0	1
σ^{10}	0	0	0	0	0	0	0	0	0	0
	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	σ^8	σ^9	σ^{10}
	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$

Algorithm for homology

Incremental computation of the homology

Input: an increasing sequence of simplicial complexes $K^1 \subset \cdots \subset K^n = K$
Output: the Betti numbers $\beta_0(K), ...\beta_d(K)$
$\beta_0 \leftarrow 0, ..., \beta_d \leftarrow 0$;
for $i \leftarrow 1$ **to** n **do**
  $d = \dim(\sigma^i)$;
  if σ^i *is positive* **then**
    $\beta_k(K^i) \leftarrow \beta_k(K^i) + 1$;
  else if $d > 0$ **then**
    $\beta_{k-1}(K^i) \leftarrow \beta_{k-1}(K^{i-1}) - 1$;

Gauss reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $i \leftarrow 1$ ȷo n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

Conclusion

We defined the standard algorithm for (persistent homology).

It works incrementally, by using the predicate 'test of positivity of a simplex'.

Using the algorithm, we have been able to compute the homology groups of the spheres, and as a consequence we proved the Invariance of Domain.

Homework: Exercises 31, 34

Facultative: Exercises 33

Conclusion

We defined the standard algorithm for (persistent homology).

It works incrementally, by using the predicate ‘test of positivity of a simplex’.

Using the algorithm, we have been able to compute the homology groups of the spheres, and as a consequence we proved the Invariance of Domain.

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Merci !