EMAp Summer Course

# Topological Data Analysis with Persistent Homology

https://raphaeltinarrage.github.io/EMAp.html

## Lesson 5: Homological algebra

Last update: February 1, 2021

# Introduction



Until here, we defined two invariants of topological spaces: *number of connected components* and *Euler characteristic.*

Today we will define a powerful invariant, *homology groups*, that already contains the two previous invariants.

Algebraic topology

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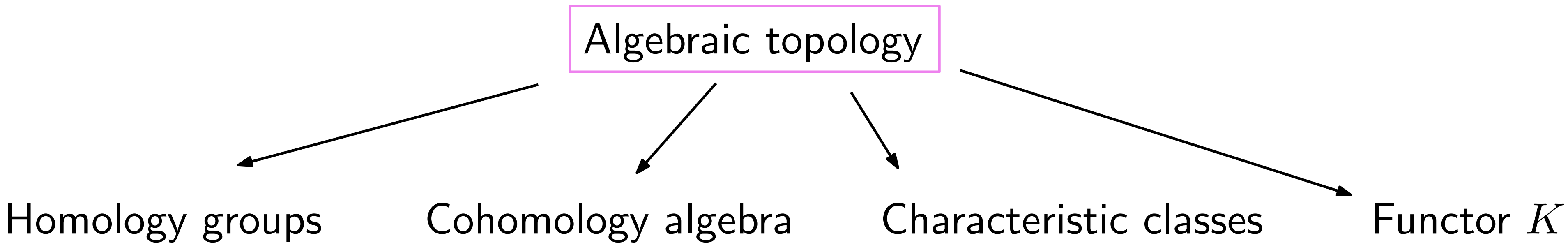

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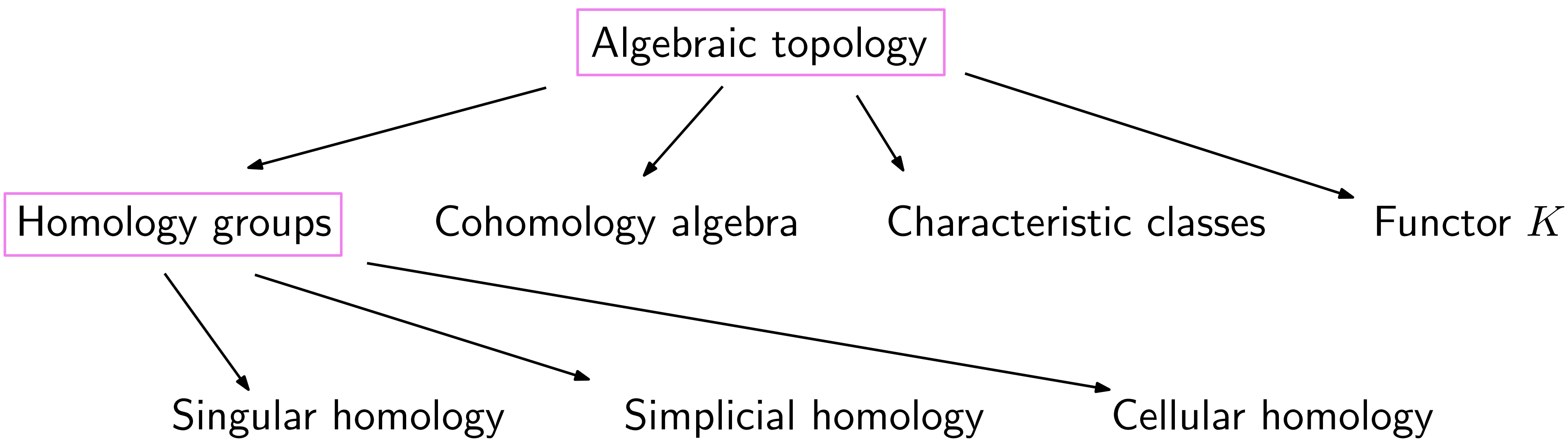

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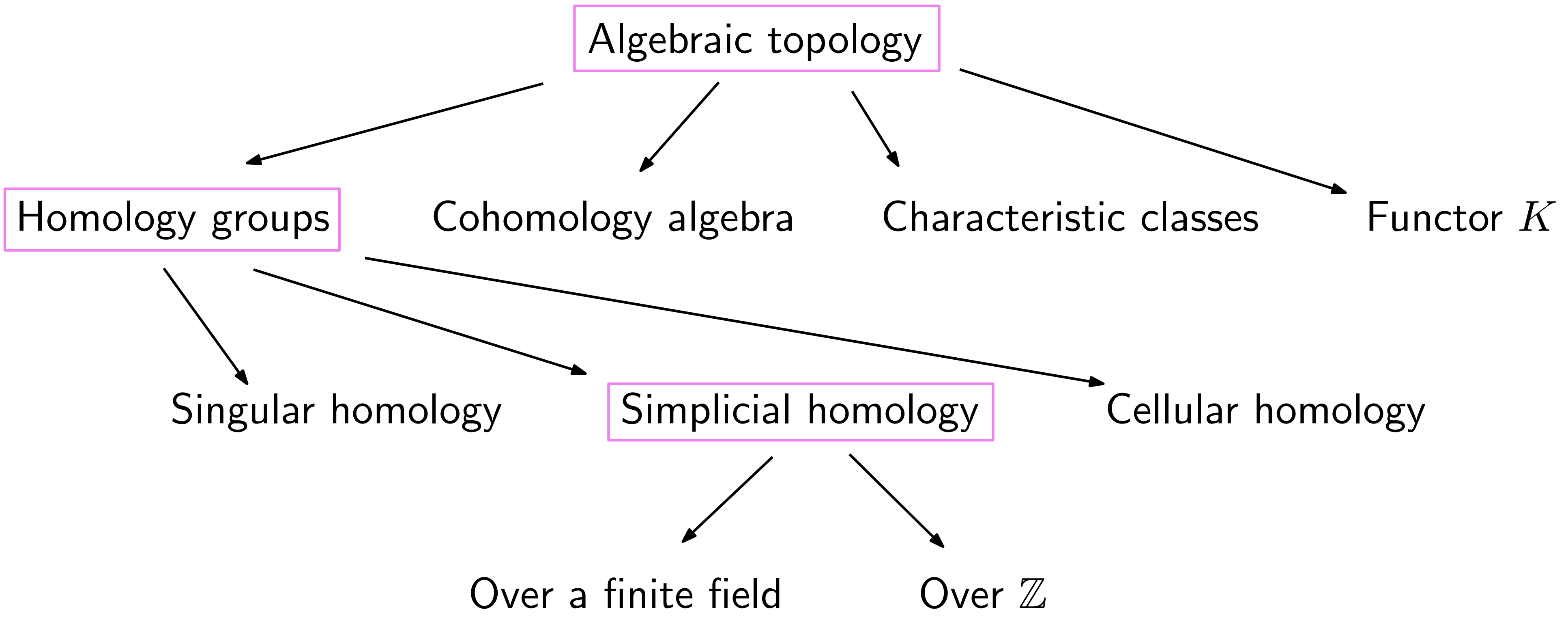

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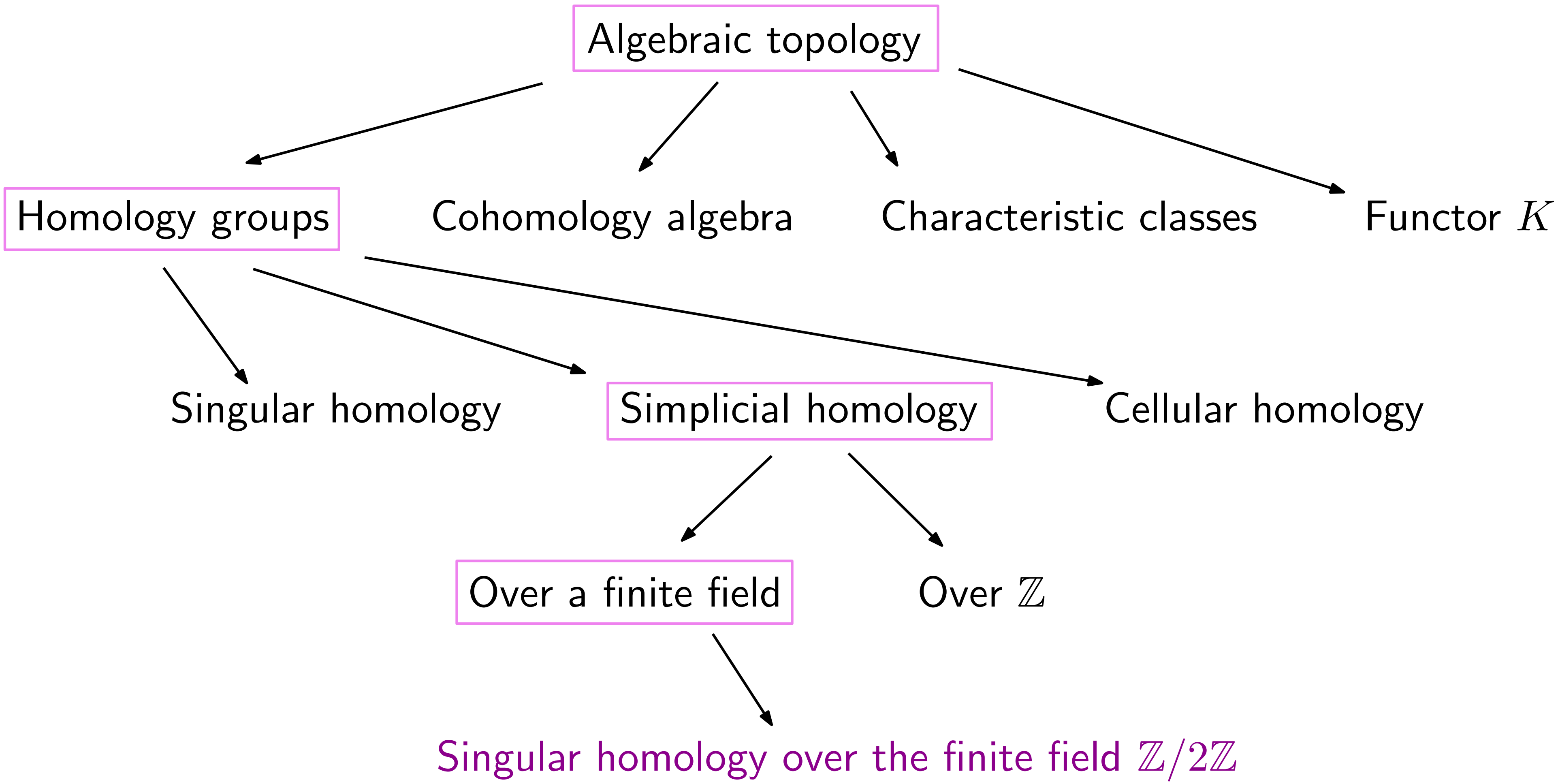

I - Reminder of algebra

II - Chains, cycles and boundaries

III - Homology groups

VI - Homology groups of topological spaces

# Groups



We recall that a *group* $(G, +)$ is a set $G$ endowed with an operation

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g + h \end{aligned}$$

such that:

- (associativity) $\forall a, b, c \in G$, $(a + b) + c = a + (b + c)$,
- (identity) $\exists\ 0 \in G, \forall a \in G, a + 0 = 0 + a = a$,
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Moreover, we say that $G$ is *commutative* if $\forall a, b \in G, a + b = b + a$. In this course, the only groups we consider will be commutative.

Example: The group of integers $(\mathbb{Z}, +)$ is commutative (1+2=2+1).

# Quotient group



A *subgroup* of $(G, +)$ is a subset $H \subset G$ such that

$$\forall a, b \in H, a + b \in H.$$

If $H$ is a subgroup of $G$, the operation $+ : G \times G \to G$ restricts to an operation $+ : H \times H \to H$, making $H$ a group on its own.

Example: For any $p \geq 1$, the set $p\mathbb{Z} = \{pn, n \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$.

Indeed, for any $m, n \in \mathbb{Z}$, $pn + pm = p(n + m) \in p\mathbb{Z}$.

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Suppose that $G$ is commutative, and that $H$ is a subgroup of $H$. We define the following equivalence relation on $G$: for all $a, b \in G$,

$$a \sim b \iff a - b \in H.$$

Denote by $G/H$ the quotient set of $G$ under this relation. For any $a \in G$, one shows that the equivalence class of $a$ is equal to $a + H = \{a + h, h \in H\}$.

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We have $a \sim b \iff a - b \in p\mathbb{Z} \iff p | a - b$.

The equivalence class of any $a \in \mathbb{Z}$ is

$$\{b \in \mathbb{Z}, p|a-b\} = \{b \in \mathbb{Z}, \exists n \in \mathbb{Z}, b = a + pn\} = a + p\mathbb{Z}$$

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Let $a_0, a_1, ..., a_n$ be a choice of representants of equivalence classes of the relation $\sim$.

The quotient set can be written as $G/H = \big\{0 + H, a_1 + H, ..., a_n + H\big\}$.

One defines a group structure $\oplus$ on $G/H$ as follows: for any $i, j \in [\![0, n]\!]$,

$$(a_i + H) \oplus (a_j + H) = (a_i + a_j) + H.$$

The group $(G/H, \oplus)$ is called the *quotient group*.

# Quotient group



Example: Consider $p\mathbb{Z} \subset \mathbb{Z}$. The quotient group $\mathbb{Z}/p\mathbb{Z}$ admits $p$ classes, with representants $a_0 = 0, ..., a_{p-1} = p-1$.
The classes are

$$p\mathbb{Z}, \quad 1 + p\mathbb{Z}, \quad 2 + p\mathbb{Z}, \quad ..., \quad p - 1 + p\mathbb{Z}.$$

The quotient group $\mathbb{Z}/p\mathbb{Z}$ can be seen as follows: its elements are

$$\{0, \quad 1, \quad 2, \quad ..., \quad p-1\}$$

and the operation is given by

$$a \oplus b = a + b \text{ (modulo } p)$$

For instance, $\mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$, and $4 + 5 = 2 \ (= 6 + 3)$.

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# The group $\mathbb{Z}/2\mathbb{Z}$



The subgroup $2\mathbb{Z} \subset \mathbb{Z}$ consists of all even numbers.
The relation $a \sim b \iff a - b \in 2\mathbb{Z}$ admits two equivalence classes:

$$2\mathbb{Z} = \{2n, n \in \mathbb{Z}\} \quad \text{and} \quad 1 + 2\mathbb{Z} = \{1 + 2n, n \in \mathbb{Z}\}$$

even numbers — odd numbers

The quotient group can be seen as the group $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with the operation

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For any $n \geq 1$, the *product group* $((\mathbb{Z}/2\mathbb{Z})^n, +)$ is the group whose underlying set is

$$(\mathbb{Z}/2\mathbb{Z})^n = \{(\epsilon_1, ..., \epsilon_n), \epsilon_1, ..., \epsilon_n \in \mathbb{Z}/2\mathbb{Z}\}$$

and whose operation is defined as

$$(\epsilon_1, ..., \epsilon_n) + (\epsilon'_1, ..., \epsilon'_n) = (\epsilon_1 + \epsilon'_1, ..., \epsilon_n + \epsilon'_n).$$

Note that the set $(\mathbb{Z}/2\mathbb{Z})^n$ has $2^n$ elements.

# Vector spaces



Let $(\mathbb{F}, +, \times)$ be a field. We recall that a vector space over $\mathbb{F}$ is a group $(V, +)$ endowed with an operation

$$\begin{aligned} \mathbb{F} \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda \cdot v \end{aligned}$$

such that

- (compatibility of multiplication) $\forall \lambda, \mu \in \mathbb{F}, \forall v \in V, \lambda \cdot (\mu \cdot v) = (\lambda \times \mu) \cdot v$,
- (identity) $\forall v \in V, 1 \cdot v = v$ where $1$ denotes the unit of $\mathbb{F}$,
- (scalar distributivity) $\forall \mu, \nu \in \mathbb{F}, \forall v \in V$, $(\lambda + \nu) \cdot v = \lambda \cdot v + \nu \cdot v$,
- (vector distributivity) $\forall \mu \in \mathbb{F}, \forall v, w \in V$, $\lambda \cdot (u + v) = \lambda \cdot v + \nu \cdot v$.

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Let $\{v_1, ..., v_n\}$ be a collection of elements of $V$. We say that it is *free* if

$$\forall \lambda_1, ..., \lambda_n \in \mathbb{F}, \sum_{1 \leq i \leq n} \lambda_i v_i = 0 \implies \lambda_1 = ... = \lambda_n = 0.$$

We say that it is *spans* $V$ if

$$\forall v \in V, \exists \lambda_1, ..., \lambda_n \in \mathbb{F}, \sum_{1 \leq i \leq n} \lambda_i v_i = v.$$

If the collection $\{v_1, ..., v_n\}$ is free and spans $V$, we say that it is a *basis*.

Proposition: If the vector space is finite, it admits a basis, and all bases have the same cardinal, called the *dimension* of $V$.

# Vector spaces



A *linear subspace* of $(V, +, \cdot)$ is a subset $W \subset V$ such that

$$\forall u, v \in W, u + v \in W \qquad \text{and} \qquad \forall v \in W, \forall \lambda \in \mathbb{F}, \lambda v \in W.$$

Just as for groups, we can define an equivalence relation $\sim$ on $V$, and a *quotient vector space* $V/W$.

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Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be two vector spaces. A linear map is a map $f \colon V \to W$ such that

$$\forall u, v \in V, f(u + v) = f(u) + f(v) \qquad \text{and} \qquad \forall v \in V, \forall \lambda \in \mathbb{F}, f(\lambda v) = \lambda \cdot f(v).$$

If $f$ is a bijection, it is called an *isomorphism*, and we say that $V$ and $W$ are *isomorphic*.

Proposition: If $(V, +, \cdot)$ is a vector space of dimension $n$, one shows that it is isomorphic to the product vector space $\mathbb{F}^n$.

# $\mathbb{Z}/2\mathbb{Z}$-vector spaces



Proposition: Le $(V, +)$ be a commutative group.
It can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure iff $\forall v \in V, v + v = 0$.

Proof: Suppose that $(V, +, \cdot)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space. For all $v \in V$, we have

$$0 = 0 \cdot v = (1 + 1) \cdot v = v + v.$$

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Proposition: Let $(V, +, \cdot)$ be a finite $\mathbb{Z}/2\mathbb{Z}$-vector space. Then there exists $n \geq 0$ such that $V$ has cardinal $2^n$, and $(V, +, \cdot)$ is isomorphic to the vector space $(\mathbb{Z}/2\mathbb{Z})^n$.

Proof: Consequence of the theory of vector spaces.

I - Reminder of algebra

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# Skeleton

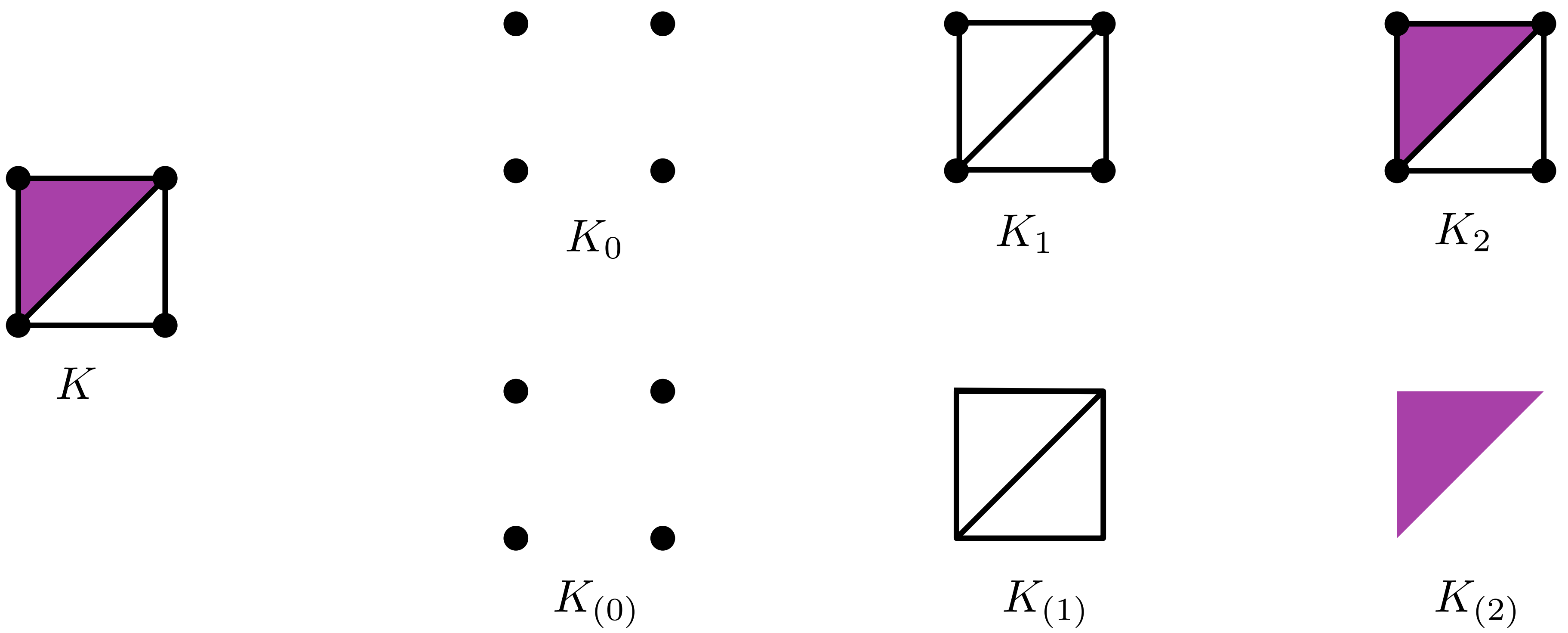


Let $K$ be a simplicial complex. For any $n \geq 0$, define the $n$-*skeleton* of $K$:

$$K_n = \{\sigma \in K, \dim(\sigma) \leq n\}$$

Also, define

$$K_{(n)} = \{\sigma \in K, \dim(\sigma) = n\}.$$



# Chains

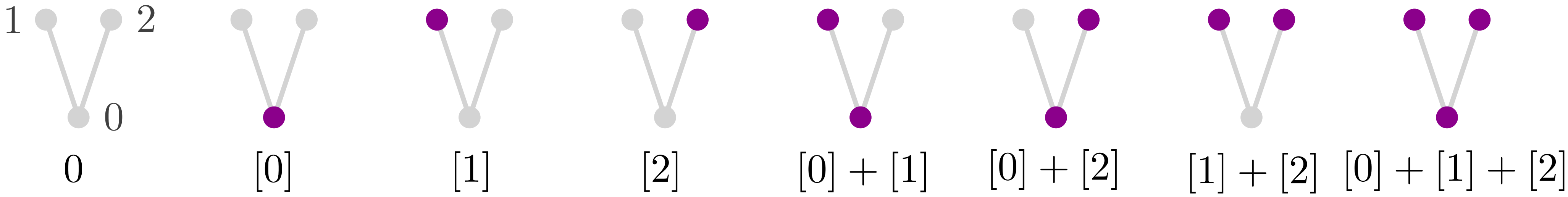


Let $n \geq 0$. The $n$-*chains* of $K$ is the set $C_n(K)$ whose elements are the formal sums

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \quad \text{where} \quad \forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}.$$

Example: Consider the simplicial complex $K = \{[0],[1],[2],[0,1],[0,2]\}$. The 0-chains $C_0(K)$ consists in 8 elements:

$$C_0(K) = \{0,\ [0],\ [1],\ [2],\ [0]+[1],\ [0]+[2],\ [1]+[2],\ [0]+[1]+[2]\}$$



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$0$ $[0]$ $[1]$ $[2]$ $[0]+[1]$ $[0]+[2]$ $[1]+[2]$ $[0]+[1]+[2]$

The $1$-chains $C_1(K)$ consists in $4$ elements

$$C_1(K) = \{0,\ [0,1],\ [0,2],\ [0,1]+[0,2]\}.$$

$0$ $[0,1]$ $[0,2]$ $[0,1]+[0,2]$

# Chains



Let $n \geq 0$. The $n$-*chains* of $K$ is the set $C_n(K)$ whose elements are the formal sums

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \quad \text{where} \quad \forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}.$$

We can give $C_n(K)$ a group structure via

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma + \sum_{\sigma \in K_{(n)}} \eta_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} (\epsilon_\sigma + \eta_\sigma) \cdot \sigma.$$

Moreover, $C_n(K)$ can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure.
To see this, remember that a group $(V, +)$ can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure if and only if $\forall v \in V, v + v = 0$. Now, observe that for any element of $C_n(K)$,

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma + \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} (\epsilon_\sigma + \epsilon_\sigma) \cdot \sigma = \sum_{\sigma \in K_{(n)}} 0 \cdot \sigma = 0.$$

Example: In the simplicial complex $K = \{[0], [1], [2], [0,1], [0,2]\}$, the sum of the $0$-chains $[0] + [1]$ and $[0] + [2]$ is $[1] + [2]$:

$$([0] + [1]) + ([0] + [2]) = [0] + [0] + [1] + [2] = [1] + [2].$$

# Boundary operator



Let $n \geq 1$, and $\sigma = [x_0, ..., x_n] \in K_{(n)}$ a simplex of dimension $n$. We define its *boundary* as the following element of $C_{n-1}(K)$:

$$\partial_n \sigma = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau$$

We can extend the operator $\partial_n$ as a linear map $\partial_n : C_n(K) \to C_{n-1}(K)$ as follows: for any element of $C_n(K)$,

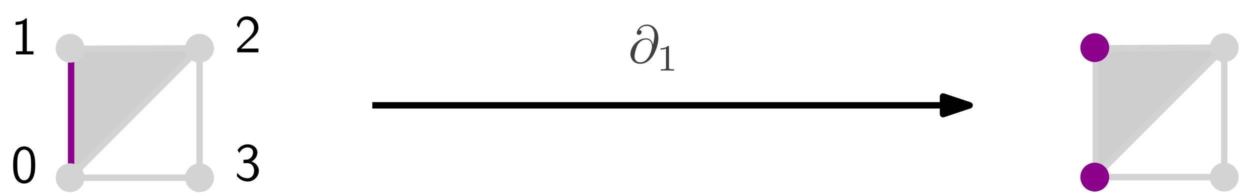
$$\partial_n \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \partial_n \sigma.$$

Example: Consider the simplicial complex

$$K = \{[0], [1], [2], [3], [0,1], [0,2], [1,2], [1,3], [2,3], [0,1,2]\}.$$

The simplex $[0,1]$ has the faces $[0]$ and $[1]$. Hence

$$\partial_1 [0,1] = [0] + [1].$$



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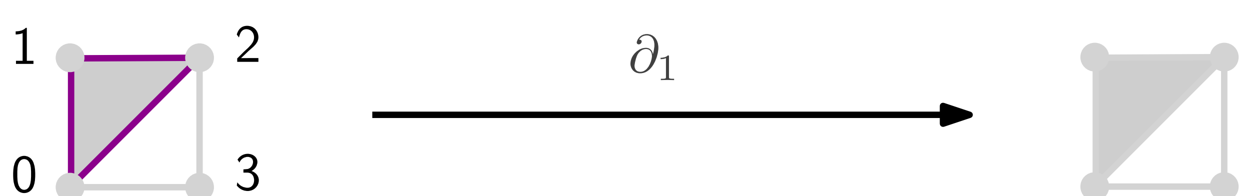
The boundary of the 1-chain $[0,1] + [1,2] + [2,0]$ is

$$\begin{aligned} \partial_1\big([0,1] + [1,2] + [2,0]\big) &= \partial_1[0,1] + \partial_1[0,2] + \partial_1[2,0] \\ &= [0] + [1] + [0] + [2] + [2] + [0] = 0 \end{aligned}$$



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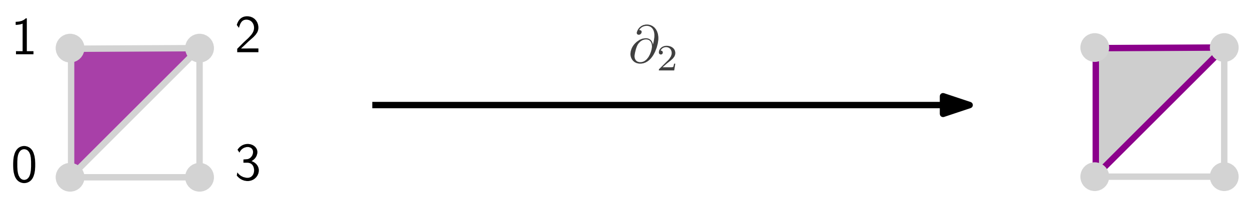
$$K = \{[0], [1], [2], [3], [0,1], [0,2], [1,2], [1,3], [2,3], [0,1,2]\}.$$

The simplex $[0,1,2]$ has the faces $[0,1]$ and $[1,2]$ and $[2,0]$. Hence

$$\partial_2 [0,1,2] = [0,1] + [1,2] + [2,0].$$



# Boundary operator



Proposition: For any $n \geq 1$, for any $c \in C_n(K)$, we have $\partial_{n-1} \circ \partial_n(c) = 0$.

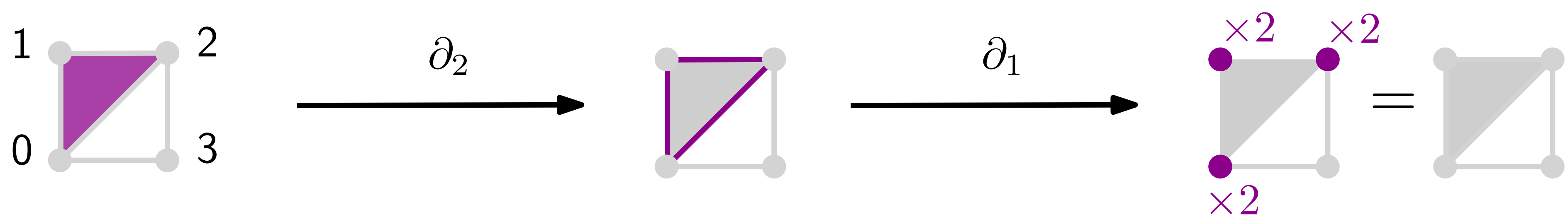
Proof: Suppose that $n \geq 2$, the result being trivial otherwise.
Since the boundary operators are linear, it is enough to prove that $\partial_{n-1} \circ \partial_n(\sigma) = 0$ for all simplex $\sigma \in K_{(n)}$.
By definition, $\partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau$, and

$$\partial_{n-1} \circ \partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \partial_{n-1}(\tau) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \sum_{\substack{\nu \subset \tau \\ |\nu| = |\tau| - 1}} \nu$$

We can write this last sum as

$$\sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \sum_{\substack{\nu \subset \tau \\ |\nu| = |\tau| - 1}} \nu = \sum_{\substack{\nu \subset \sigma \\ |\nu| = |\sigma| - 2}} \alpha_\nu \nu$$

where $\alpha_\nu = \{\tau \subset \sigma, |\tau| = |\sigma| - 1, \nu \subset \tau\}$.
It is easy to see that for every $\nu$ such that $\dim \nu = \dim \tau - 2$, we have $\alpha_\nu = 2 = 0$.



# Cycles and boundaries



Let $n \geq 0$. We have a triplet of vector spaces

$$C_{n+1}(K) \xrightarrow{\partial n+1} C_n(K) \xrightarrow{\partial n} C_{n-1}(K).$$

The maps $\partial_{n+1}$ and $\partial_n$ are linear maps, and we can consider their kernel and image.

Definition: We define:

- The $n$-*cycles*: $Z_n(K) = \mathrm{Ker}(\partial_n)$,
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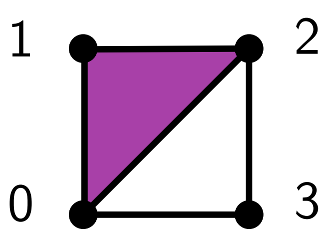
Example:
Consider the following simplicial complex




The chains
0, $[0,1]+[1,2]+[0,2]$, $[0,2]+[2,3]+[0,3]$ and $[0,1]+[1,2]+[2,3]+[0,3]$
are 1-cycles.
For instance,

$$\partial_1([0,1]+[1,2]+[0,2]) = [0]+[1]+[1]+[2]+[0]+[2] = 0.$$

Moreover, the chains

$$\partial_2(0) = 0 \quad \text{and} \quad \partial_2([0,1,2]) = [0,1]+[0,2]+[1,2].$$

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Definition: We say that two chains $c, c' \in C_n(K)$ are *homologous* if there exists $b \in B_n(K)$ such that $c = c' + b$.

⟶ two chains are homologous if they are equal up to a boundary

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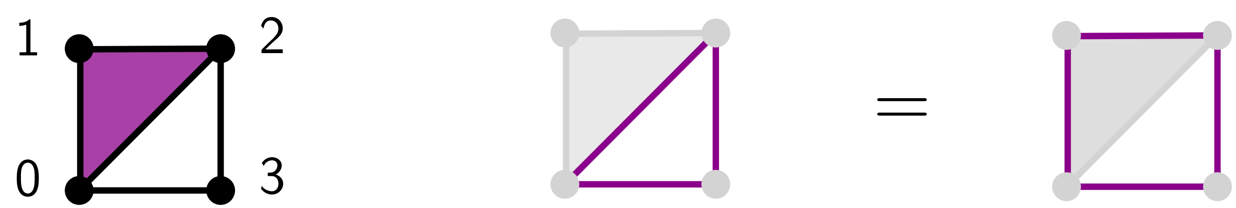


The chains $[0,2] + [2,3] + [0,3]$ and $[0,1] + [1,2] + [2,3] + [0,3]$ are homologous. Indeed,

$$[0,2] + [2,3] + [0,3] = [0,1] + [1,2] + [2,3] + [0,3] + [0,1] + [0,2] + [1,2].$$

# Cycles and boundaries



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- The $n$-*cycles*: $Z_n(K) = \mathrm{Ker}(\partial_n)$,
- The $n$-*boundaries*: $B_n(K) = \mathrm{Im}(\partial_{n+1})$.

Proposition: We have $B_n(K) \subset Z_n(K)$.

Proof: Let $b \in B_n(K)$ be a boundary.
By definition, there exists $c \in C_{n+1}(K)$ such that $b = \partial_{n+1} c$.
Using $\partial_n \partial_{n+1} = 0$, we get

$$\partial_n b = \partial_n \partial_{n+1} c = 0,$$

hence $b \in Z_n(K)$.

I - Reminder of algebra

II - Chains, cycles and boundaries

III - Homology groups

VI - Homology groups of topological spaces

# Homology groups



In the previous subsection, we have defined a sequence of vector spaces, connected by linear maps

$$\ldots \to C_{n+1}(K) \to C_n(K) \to C_{n-1}(K) \to \ldots$$

and for every $n \geq 0$, we have defined the cycles and the boundaries $Z_n(K)$ and $B_n(K)$.
Since $B_n(K) \subset Z_n(K)$, we can see $B_n(K)$ as a linear subspace of $Z_n(K)$.
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Proposition: $\dim H_n(K) = \dim B_n(K) - \dim Z_n(K)$.

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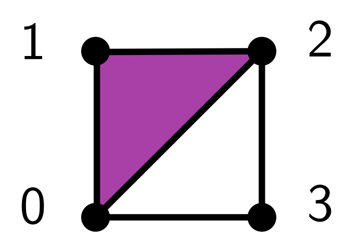
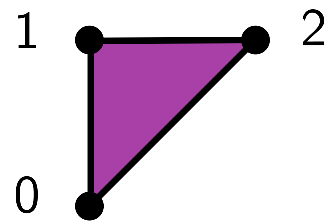
Exemple: Consider the simplicial complex




As we have seen, $Z_1(K)$ has cardinal $4$, and $B_1(K)$ cardinal $2$. We deduce that $\dim Z_1(K) = 2$, $\dim B_1(K) = 1$, hence $\dim H_1(K) = 2 - 1 = 1$.

In other words, we have an isomorphism $H_1(K) \simeq \mathbb{Z}/2\mathbb{Z}$. Also, $\beta_1(K) = 1$.

Exercise: Consider the simplicial complex

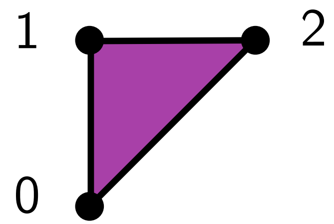



Compute its $0$-boundaries and $0$-cycles. Deduce $\beta_0(K)$.

# Homology groups



Exercise: Consider the simplicial complex




Compute its $0$-boundaries and $0$-cycles. Deduce $\beta_0(K)$.

We have:

$Z_0(K) = \{[0], [1], [2], [0] + [1], [0] + [2], [1] + [2], [0] + [1] + [2]\}$

$B_0(K) = \{0, [0] + [1], [1] + [2], [0] + [2]\}$.

We deduce that $\dim Z_0(K) = 3$, $\dim B_0(K) = 2$, and

$$\dim H_0(K) = 3 - 2 = 1.$$

I - Reminder of algebra

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# Invariant property

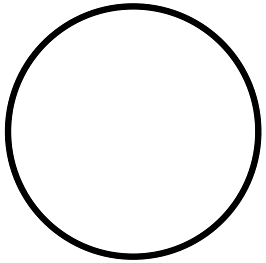
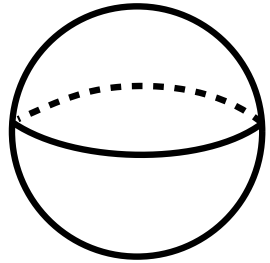
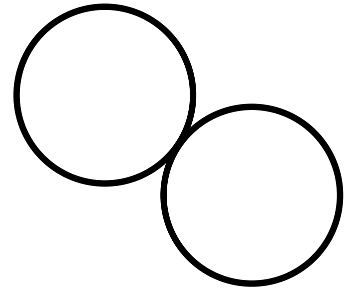
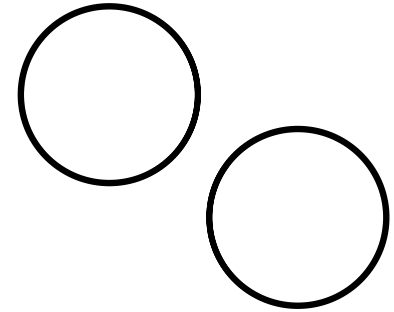


Definition: The homology groups of a topological space are the homology groups of any triangulation of it. We define its Betti numbers similarly.

Proposition: If $X$ and $Y$ are two homotopy equivalent topological spaces, then for any $n \geq 0$ we have isomorphic homology groups $H_n(X) \simeq H_n(Y)$. As a consequence, $\beta_n(X) = \beta_n(Y)$.

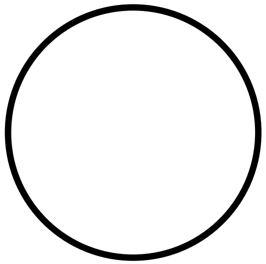
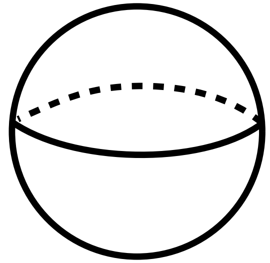
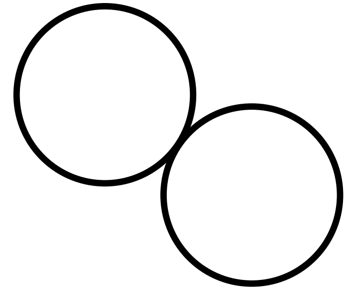
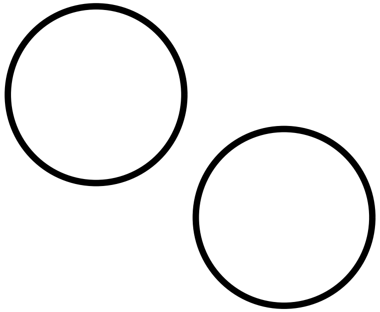
# Examples



| $X$ |  |  |  |  |
|---|---|---|---|---|
| $H_0(X)$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| $\beta_0(X)$ | 1 | 1 | 1 | 2 |
| $H_1(X)$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| $\beta_1(X)$ | 1 | 0 | 2 | 2 |
| $H_2(X)$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 |
| $\beta_2(X)$ | 0 | 1 | 0 | 0 |

# Examples



| | | | | |
|---|---|---|---|---|
| $X$ |  |  |  |  |
| $H_0(X)$ | $\mathbb{Z}/2\mathbb{Z}$ | | | $2\mathbb{Z})^2$ |
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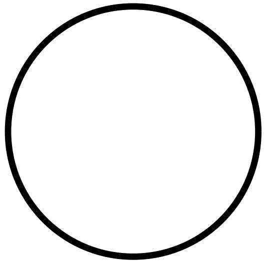
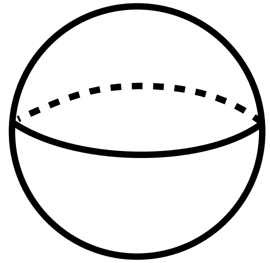
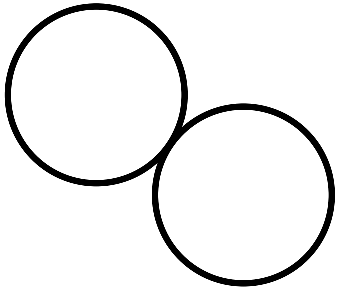
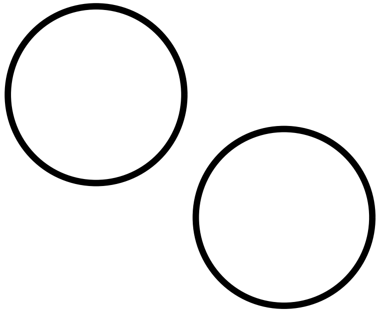
| $H_1(X)$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ |
| $\beta_1(X)$ | 1 | 0 | 2 | 2 |
| $H_2(X)$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 |
| $\beta_2(X)$ | 0 | 1 | 0 | 0 |

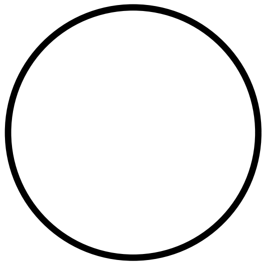
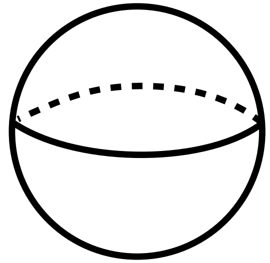
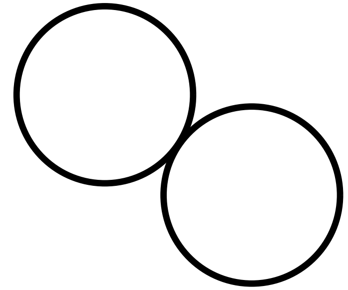
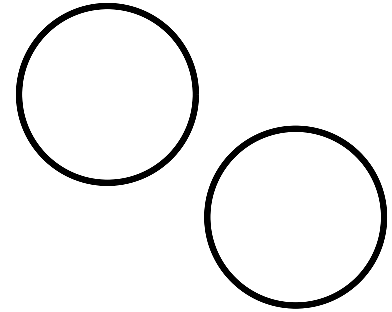
Number of connected components

# Examples

| | | | | |
|---|---|---|---|---|
| $X$ |  |  |  |  |
| $H_0(X)$ | $\mathbb{Z}/2\mathbb{Z}$ | Number of connected components | | $2\mathbb{Z})^2$ |
| $\beta_0(X)$ | 1 | 1 | 1 | 2 |
| $H_1(X)$ | $\mathbb{Z}/2\mathbb{Z}$ | Number of holes | | $2\mathbb{Z})^2$ |
| $\beta_1(X)$ | 1 | 0 | 2 | 2 |
| $H_2(X)$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 |
| $\beta_2(X)$ | 0 | 1 | 0 | 0 |

# Examples



| $X$ |  |  |  |  |
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| $\beta_1(X)$ | 1 | 0 | 2 | 2 |
| $H_2(X)$ | 0 | Number of cavities | | 0 |
| $\beta_2(X)$ | 0 | 1 | 0 | 0 |

# Conclusion

We defined the homology groups of simplicial complexes.

We defined the homology groups of topological spaces via triangulations.

This is an invariant of homotopy equivalence.

Homeworks for next week: Exercises 29, 30

Facultative exercise: Exercises 27, 28

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Merci !