EMAp Summer Course

Topological Data Analysis with Persistent Homology

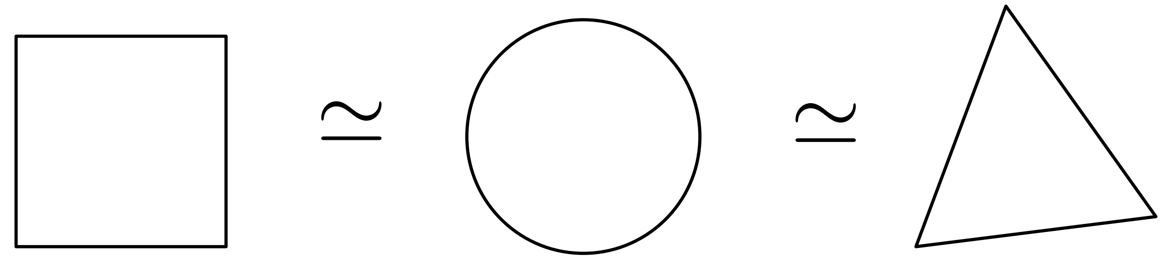
https://raphaeltinarrage.github.io/EMAp.html

Lesson 3: Homotopies

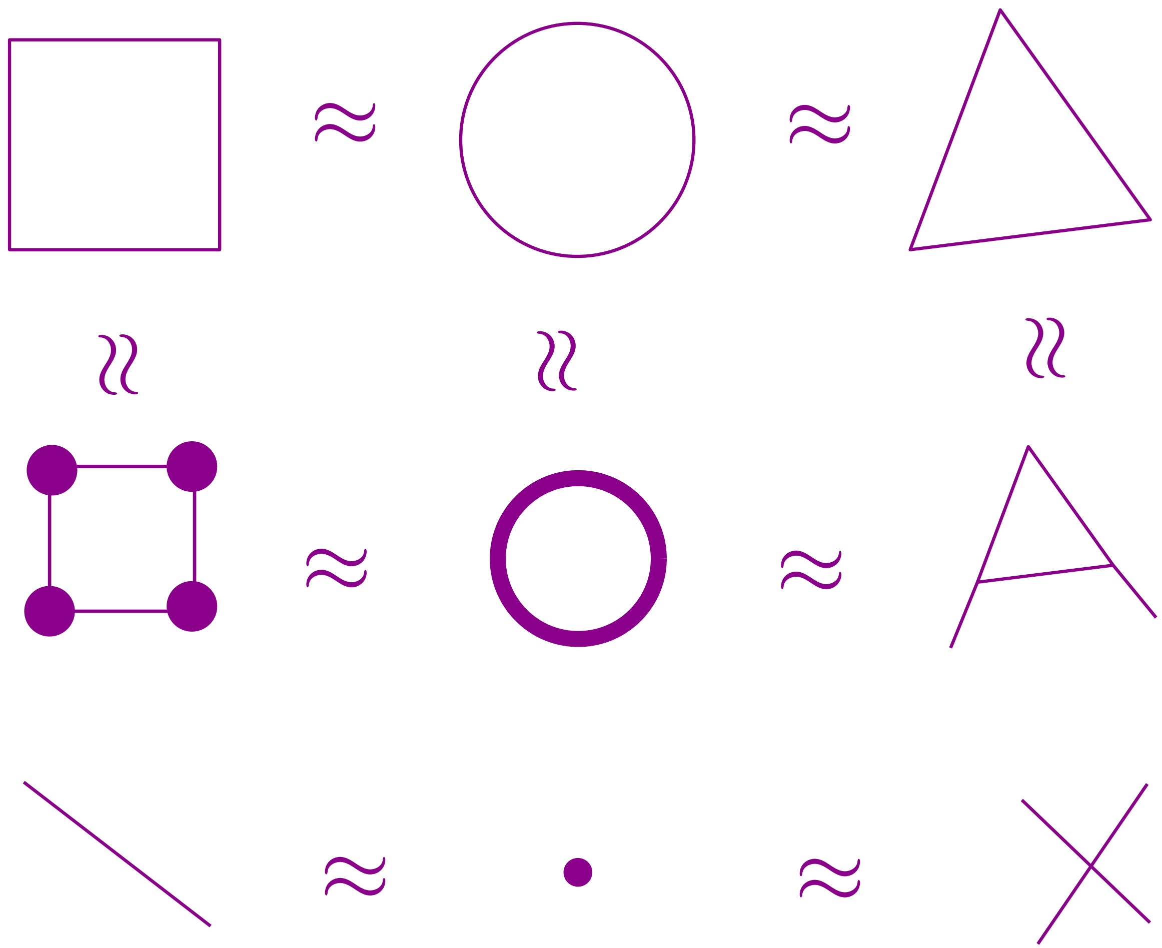
Last update: January 28, 2021

Introduction

Homeomorphism equivalence



Homotopy equivalence



I - Homotopy equivalence between maps

II - Homotopy equivalence between topological spaces

III - Link with homeomorphic spaces

VI - Invariants

Definition

Definition: Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces, and $f, g \colon X \to Y$ two continuous maps. A *homotopy* between f and g is a map $F \colon X \times [0,1] \to Y$ such that:

- $F(\cdot, 0)$ is equal to f,
- $F(\cdot, 1)$ is equal to g,
- $F \colon X \times [0,1] \to Y$ is continuous.

If such a homotopy exists, we say that the maps f and g are *homotopic*.

For any $t \in [0,1]$, the notation $F(\cdot, t)$ refers to the map

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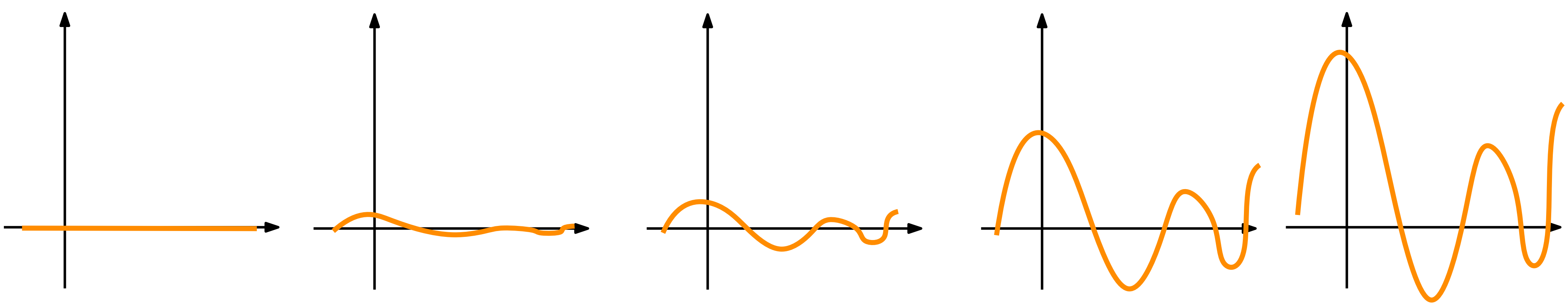
Example: Homotopy $F \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ between $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$.



$F(\cdot, 0) = f$ $\quad$ $F(\cdot, 0.2)$ $\quad$ $F(\cdot, 0.5)$ $\quad$ $F(\cdot, 0.6)$ $\quad$ $F(\cdot, 1) = g$

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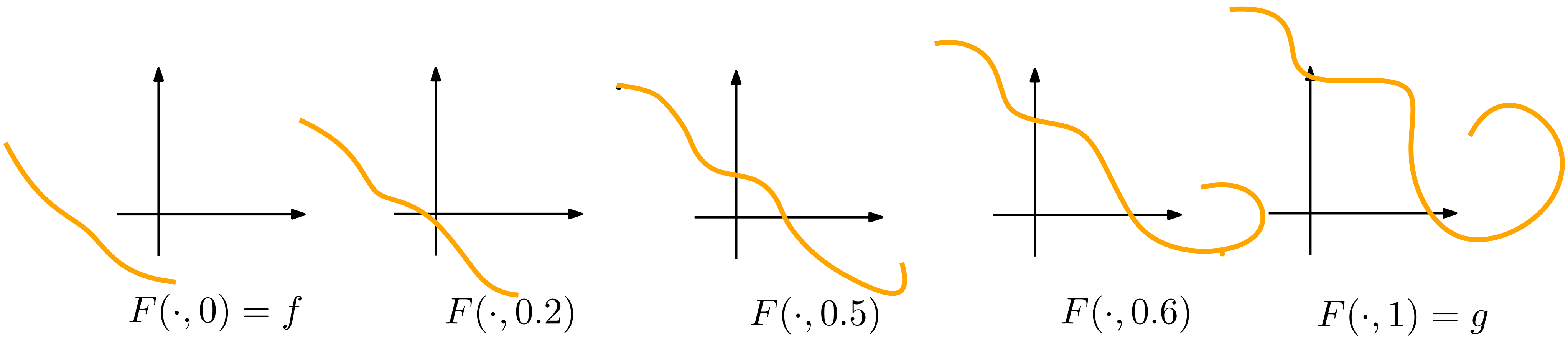
Example: Homotopy $F\colon [0,1] \times [0,1] \to \mathbb{R}^2$ between $f\colon [0,1] \to \mathbb{R}^2$ and $g\colon [0,1] \to \mathbb{R}^2$.



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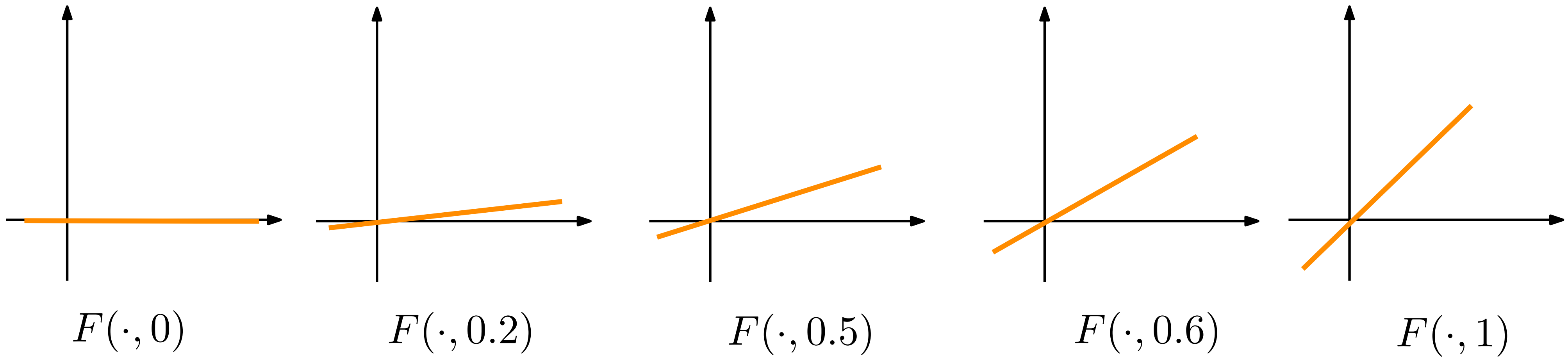
Example: Let $X = Y = [-1, 1]$ endowed with the Euclidean topology, and consider the maps $f, g\colon X \to Y$ defined as

$$f\colon x \mapsto 0$$
$$g\colon x \mapsto x$$

Let us prove that they are homotopic. Consider the map

$$\begin{aligned} F\colon X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto tx \end{aligned}$$

We see that $F(\cdot, 0)\colon x \mapsto 0$ is equal to f, and $F(\cdot, 1)\colon x \mapsto x$ is equal to g. Moreover, F is continuous. Hence, F is an homotopy between f and g. Thus these two maps are homotopic.



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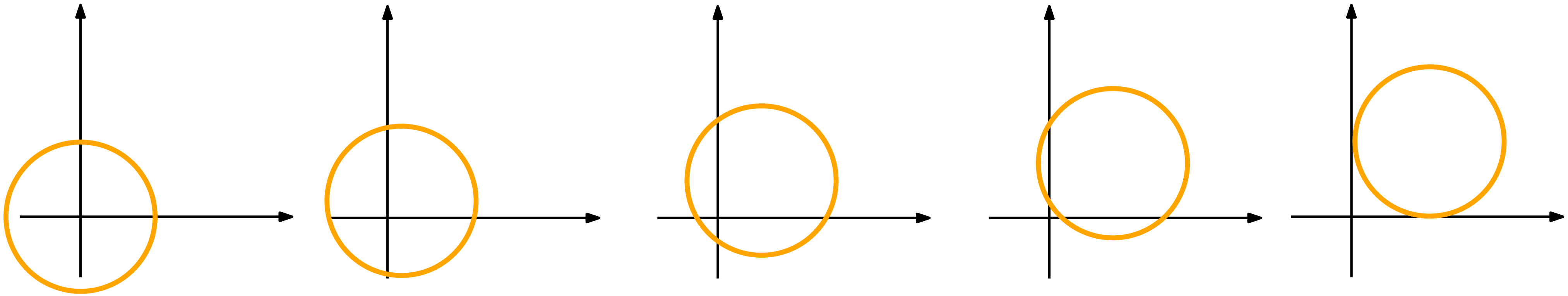

$F(\cdot, 0)$ $F(\cdot, 0.2)$ $F(\cdot, 0.5)$ $F(\cdot, 0.6)$ $F(\cdot, 1)$

Example: The map

$$\begin{aligned} F\colon\ \mathbb{S}_1 \times [0,1] &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto (\cos(\theta) + t, \sin(\theta) + t) \end{aligned}$$

is a homotopy between the maps $f\colon\ \mathbb{S}^1 \to \mathbb{R}^2$ and $g\colon\ \mathbb{S}^1 \to \mathbb{R}^2$ defined as

$$f\colon \theta \mapsto (\cos(\theta), \sin(\theta)) \qquad \text{and} \qquad g\colon \theta \mapsto (\cos(\theta) + 1, \sin(\theta) + 1)$$

(Notation: $\mathbb{S}_1$ denotes the unit circle of $\mathbb{R}^2$.)

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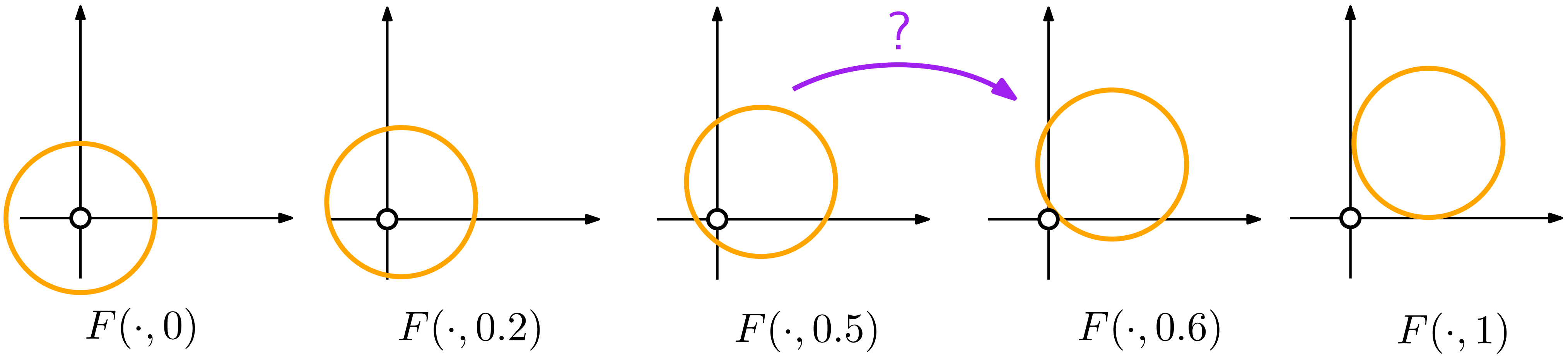

Non-example: Between $\mathbb{S}_1$ and $\mathbb{R}^2 \setminus \{(0,0)\}$, the plane without the origin, there is no homotopy between the maps

$$f\colon \theta \mapsto (\cos(\theta), \sin(\theta)) \qquad \text{and} \qquad g\colon \theta \mapsto (\cos(\theta)+1, \sin(\theta)+1)$$

The homotopy F would pass through the point $(0,0)$ at some point, which is impossible.

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Trivial maps

From a homotopic point a view, a *trivial map* is a map that is homotopic to a constant map.

Proposition: Let $(X, \mathcal{T})$ be a topological space and $f\colon X \to \mathbb{R}^n$ a continuous map. Then f is homotopic to a constant map.

Proof: Consider the continuous application

$$\begin{aligned} F\colon X \times [0,1] &\longrightarrow \mathbb{R}^n \\ (x,t) &\longmapsto tf(x) \end{aligned}$$

We have that $F(\cdot, 1) = f$, and $F(\cdot, 0)\colon x \mapsto 0$ is a constant map.
Hence F is a homotopy between f and the constant map $\begin{aligned} g\colon X &\to \mathbb{R}^n \\ x &\mapsto 0 \end{aligned}$

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Exercise: Let $f\colon \mathbb{R}^n \to X$ be a continuous map. Then f is homotopic to a constant map.

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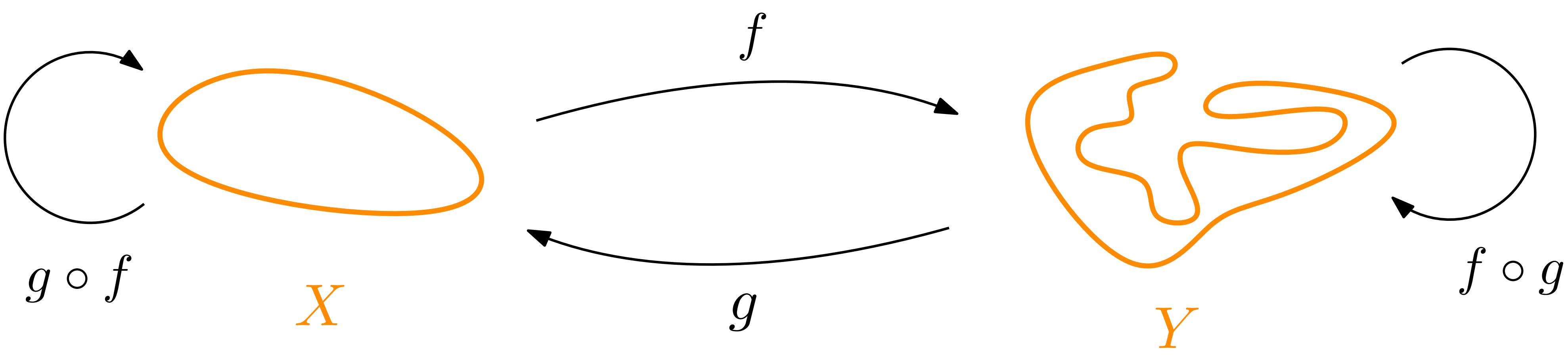

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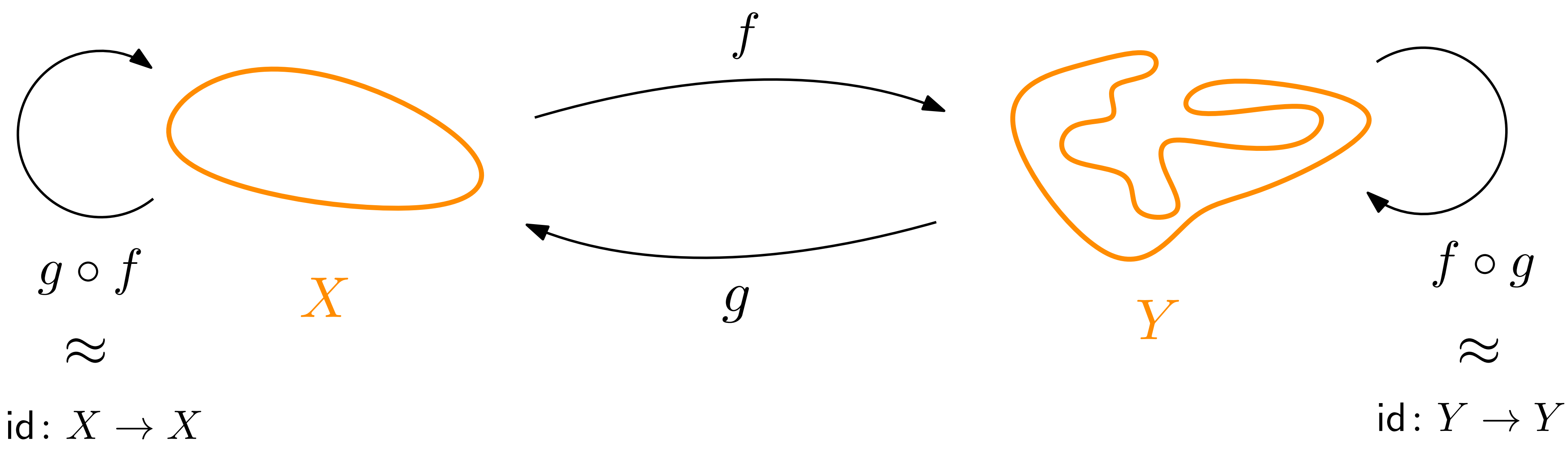
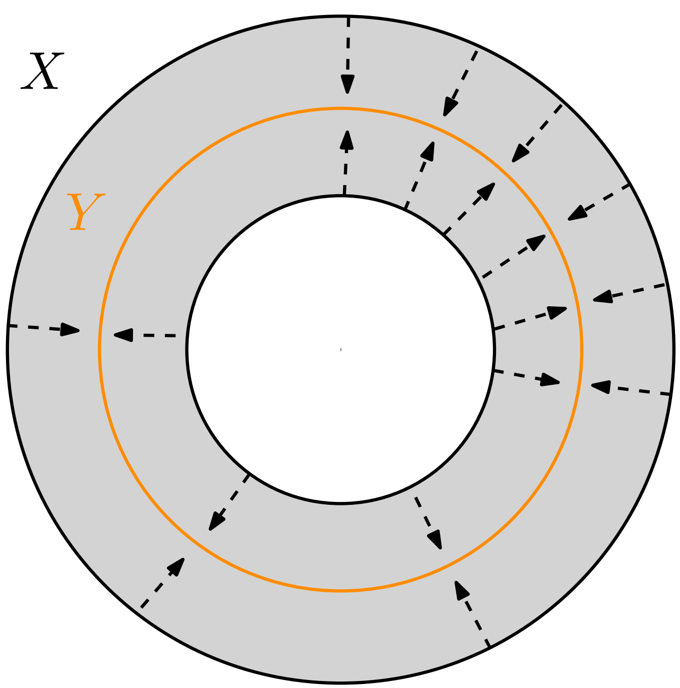


Deformation retractions

Determining whether two topological spaces are homotopy equivalent may be difficult. When one is a subset of the other, we have a handy tool:

Definition: Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $\mathcal{T}_{|Y}$.
A *retraction* is a continuous map $r\colon X \to X$ such that $\forall x \in X, r(x) \in Y$ and $\forall y \in Y, r(y) = y$.
A *deformation retraction* is a homotopy $F\colon X \times [0,1] \to Y$ between the identity map $\mathrm{id}\colon X \to X$ and a retraction $r\colon X \to Y$.



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Proposition: If a deformation retraction exists, then X and Y are homotopic equivalent.

Proof: Let $r\colon X \to Y$ denote the retraction, and consider the inclusion map $i\colon Y \to X$. Let us prove that r, i is a homotopy equivalence.

First, let us prove that $i \circ r\colon X \to X$ is homotopic to the identity map $\mathrm{id}\colon X \to X$. This is clear because $i \circ r = r$, and r is homotopic to the identity by definition of a deformation retraction.

Second, let us prove that $r \circ i\colon Y \to Y$ is homotopic to the identity map $\mathrm{id}\colon Y \to Y$. This is obvious because $r \circ i = \mathrm{id}$ by definition of a retraction.

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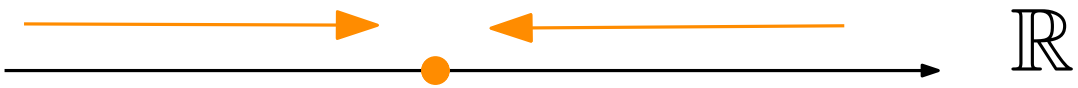
Example: For any $n \geq 1$, the Euclidean space $\mathbb{R}^n$ is homotopy equivalent to the point $\{0\} \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r\colon \mathbb{R}^n &\longrightarrow \{0\} \\ x &\longmapsto 0 \end{aligned}$$

It is homotopic to the identity $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ via the deformation retraction

$$\begin{aligned} F\colon \mathbb{R}^n \times [0,1] &\longrightarrow \mathbb{R}^n \\ (x,t) &\longmapsto (1-t)x \end{aligned}$$

Indeed, we have $F(\cdot, 0) = \mathrm{id}$ and $F(\cdot, 1) = r$.



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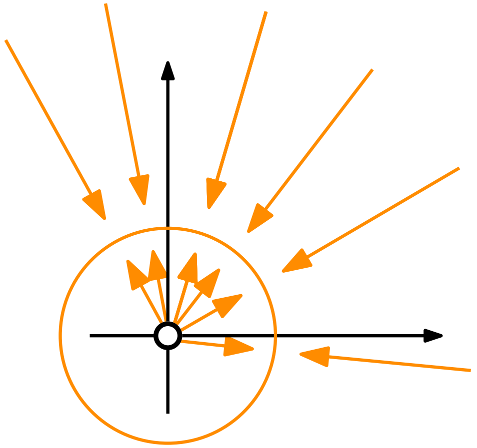
Example: For any $n \geq 1$, the Euclidean space without origin, $\mathbb{R}^n \setminus \{0\}$, is homotopy equivalent to the sphere $\mathbb{S}(0,1) \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r\colon \mathbb{R}^n \setminus \{0\} &\longrightarrow \mathbb{S}(0,1) \\ x &\longmapsto \frac{x}{\|x\|} \end{aligned}$$

It is homotopic to the identity $\mathrm{id}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ via the deformation retraction

$$\begin{aligned} F\colon (\mathbb{R}^n \setminus \{0\}) \times [0,1] &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ (x,t) &\longmapsto \left(1 - t + \frac{t}{\|x\|}\right) x \end{aligned}$$



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Homotopy equivalence relation

Let us denote $X \approx Y$ if the two topological spaces X and Y are homotopic equivalent.

Being homotopic equivalent is an *equivalence relation*. That is:

- *(Reflexivity)* $X \approx X$
- *(Symmetry)* $X \approx Y \implies Y \approx X$.
- *(Transitivity)* $X \approx Y$ and $Y \approx Z \implies X \approx Z$.

We can classify topological spaces according to this relation, and obtain *classes of homotopy equivalence*:

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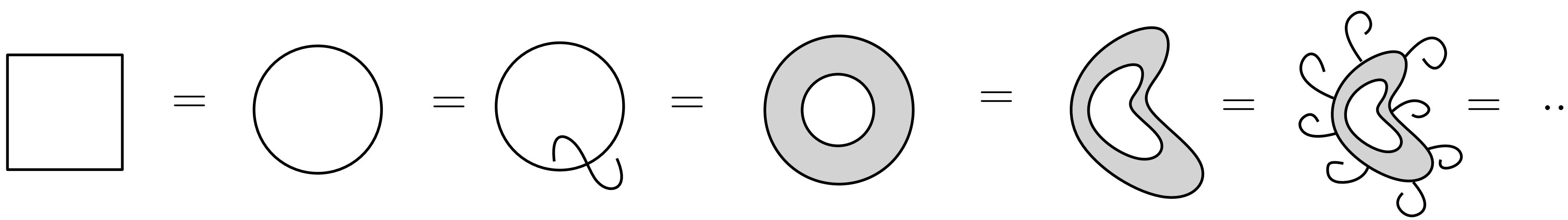

the class of circles

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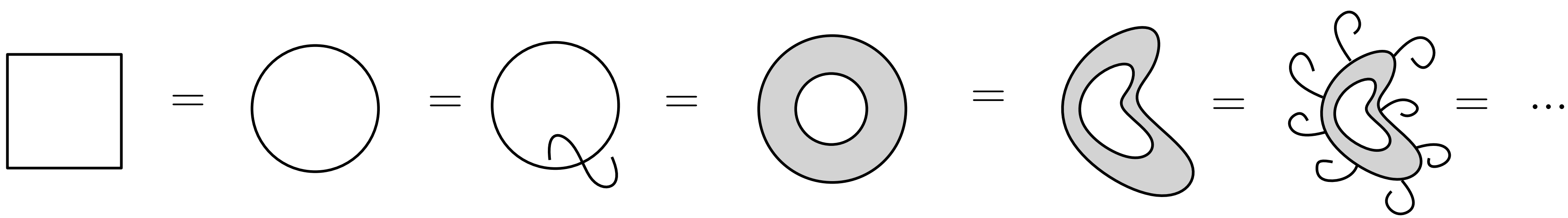
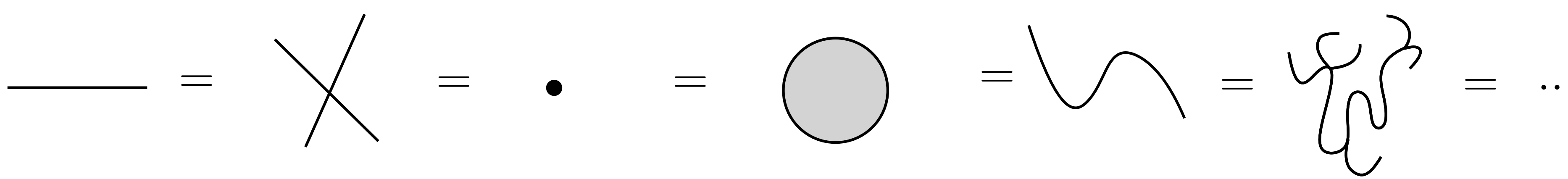


the class of circles



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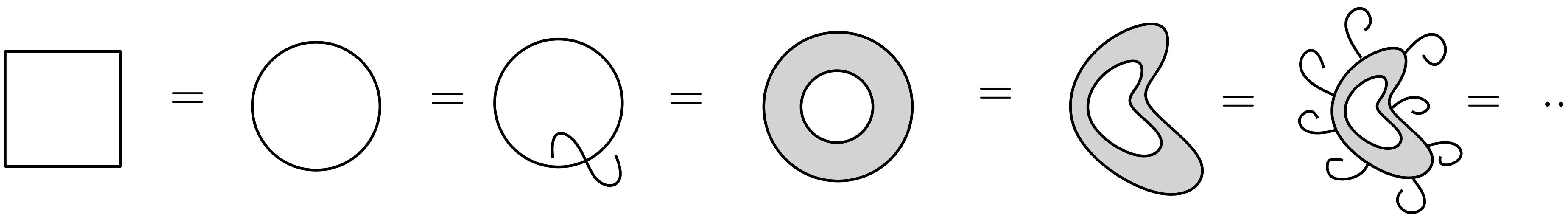

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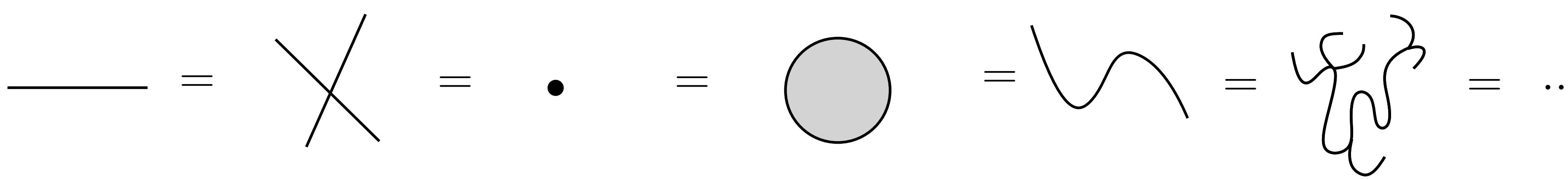

the class of points

the class of spheres, the class of torii, the class of Klein bottles, ...

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Homeomorphic implies homotopic

Proposition: Let X, Y be two topological spaces. If they are homeomorphic, then they are homotopic equivalent.

In other words:

$$X \simeq Y \implies X \approx Y.$$

Consequence: In order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic.

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Proof: Let $h\colon X \to Y$ be a homeomorphism.

We have to build a homotopy equivalence $f\colon X \to Y$, $g\colon Y \to X$.

Define $f = h$ and $g = h^{-1}$. Let us show $g \circ f$ is homotopic to $\mathrm{id}\colon X \to X$ and $f \circ g$ is homotopic to $\mathrm{id}\colon Y \to Y$.

We have $g \circ f = h^{-1} \circ h = \mathrm{id}$. But $\mathrm{id}\colon X \to X$ is homotopic to $\mathrm{id}\colon X \to X$ (any map is homotopic to itsef.)

Similarly, $f \circ g = h \circ h^{-1} = \mathrm{id}$, and $\mathrm{id}\colon Y \to Y$ is homotopic to $\mathrm{id}\colon Y \to Y$.

Conclusion: f, g is a homotopy equivalence between X and Y, hence X and Y are homotopy equivalent.

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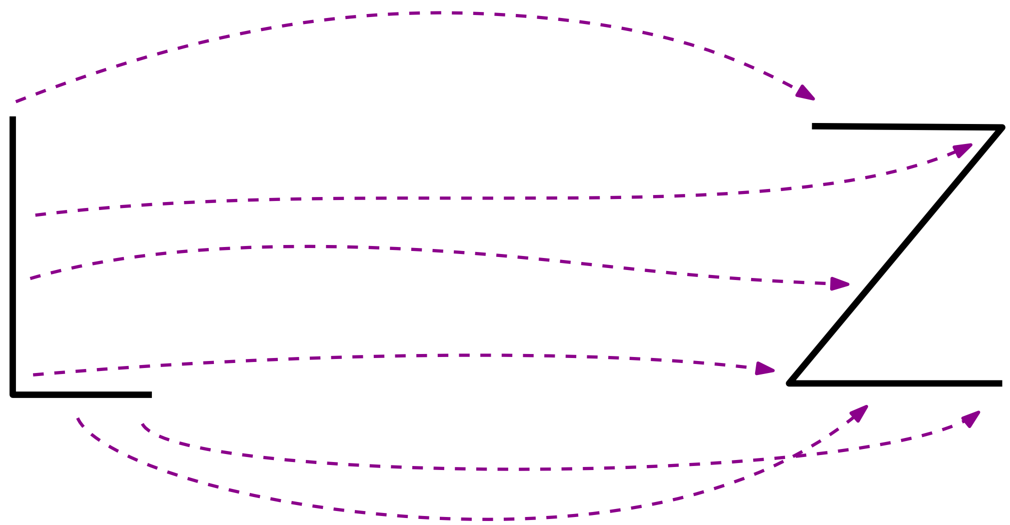
Example: The letter L and the letter Z are homeomorphic:



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Consequence: In order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic.

This strategy does not always work: some spaces are homotopy equivalent but not homeomorphic!

This is the case for $\mathbb{R}^n$ and $\{0\}$ for instance.

Exercise

Consider the following letters of the alphabet, endowed with the subspace topology induced from $\mathbb{R}^2$:

A B C D E F

Classify them into homotopy equivalence classes, then classify them into homeomorphism equivalence classes.

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Homotopy equivalence classes:

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$\longrightarrow$ B ($\approx$ ○○)

$\longrightarrow$ C $\approx$ E $\approx$ F ($\approx$ •)

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Classify them into homotopy equivalence classes, then classify them into homeomorphism equivalence classes.

Homotopy equivalence classes:

- $\longrightarrow$ $\mathsf{A} \approx \mathsf{D}$ $\quad (\approx \bigcirc)$
- $\longrightarrow$ B $\quad (\approx \bigcirc\!\bigcirc)$
- $\longrightarrow$ $\mathsf{C} \approx \mathsf{E} \approx \mathsf{F}$ $\quad (\approx \bullet)$

Homotopy equivalence classes:

- $\longrightarrow$ A
- $\longrightarrow$ C
- $\longrightarrow$ $\mathsf{E} \simeq \mathsf{F}$
- $\longrightarrow$ B
- $\longrightarrow$ D

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Number of connected components

Until here, we studied two quantities associated to topological spaces: *number of connected components* and *dimension*.

Proposition: Two homotopy equivalent topological spaces admit the same number of connected components.

Proof: Let X, Y be two topological spaces, and $f\colon X \to Y, g\colon Y \to X$ a homotopy equivalence.

Let $F\colon X \times [0,1] \to X$ be a homotopy between $g \circ f$ and $\mathrm{id}\colon X \to X$.
Let $x \in X$, and O the connected component of x.

The space $O \times [0,1]$ is connected. Hence its image $F(O \times [0,1]) \subset X$ is connected too.

Moreover, $O = F(O \times \{1\}) \subset F(O \times [0,1])$.
Hence $F(O \times [0,1])$ is a connected subset of X that contains O, and we deduce that $O = F(O \times [0,1])$.

Last, notice that

$$g \circ f(O) = F(O \times \{0\}) \subset F(O \times [0,1]) = O.$$

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Let $x \in X$, and O the connected component of x.
We have $g \circ f(O) \subset O$.

Suppose that X admits n connected components $O_1, ..., O_n$, and that Y admits m of them.

By contradiction, suppose that $m < n$. This implies that we have two components O_i, O_j such that $f(O_i)$ and $f(O_j)$ are included in the same connected component O' of Y.
Hence $g \circ f(O_i)$ and $g \circ f(O_j)$ are included in a common connected component of X.
This is absurd because $g \circ f(O_i) \subset O_i$ and $g \circ f(O_j) \subset O_j$.

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We have shown that $m \geq n$.

By exchanging the roles of X and Y in the whole reasonning, we obtain that $m \leq n$. We deduce that $m = n$.

Number of connected components

Until here, we studied two quantities associated to topological spaces: *number of connected components* and *dimension*.

Proposition: Two homotopy equivalent topological spaces admit the same number of connected components.

In other words, *number of connected components* is an invariant of homotopy equivalence.

This allows to show that two spaces are not equivalent.

Example: For any $n, m \geq 0$ such that $n \neq m$, the subspaces $\{1, ..., n\}$ and $\{1, ..., m\}$ of $\mathbb{R}$ are not homotopic equivalent.

Indeed, the first one admits n connected components, and the second one m components.

Dimension

On the other hand, dimension is **not** an invariant of homotopy equivalence.

Indeed, some homotopic equivalent spaces have different dimensions.

This is the case, for instance, with all the Euclidean spaces $\mathbb{R}^n$, $n \geq 0$. They are all homotopic equivalent, but all with different dimensions.

Conclusion

We learnt to look at topological spaces from a homotopic-equivalence perspective.

This is a weaker notion than homeomorphism-equivalence.

Between the quantities, *number of connected components* and *dimension*, ony one is invariant for the homotopic-equivalence relation.

Homework for tomorrow: Exercises 12 and 16

Facultative exercise: Exercises 13 and 14

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We learnt to look at topological spaces from a homotopic-equivalence perspective.

This is a weaker notion than homeomorphism-equivalence.

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Obrigado!