EMAp Summer Course

Topological Data Analysis with Persistent Homology

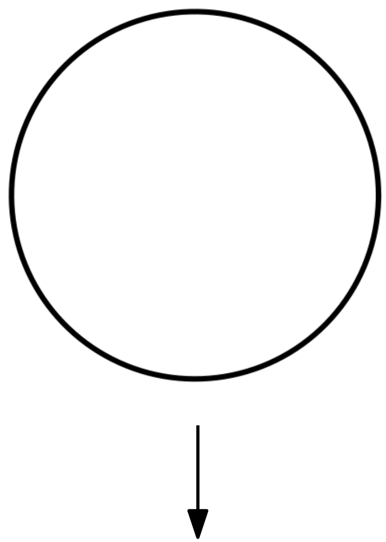
https://raphaeltinarrage.github.io/EMAp.html

Lesson 10: Stability of persistence modules

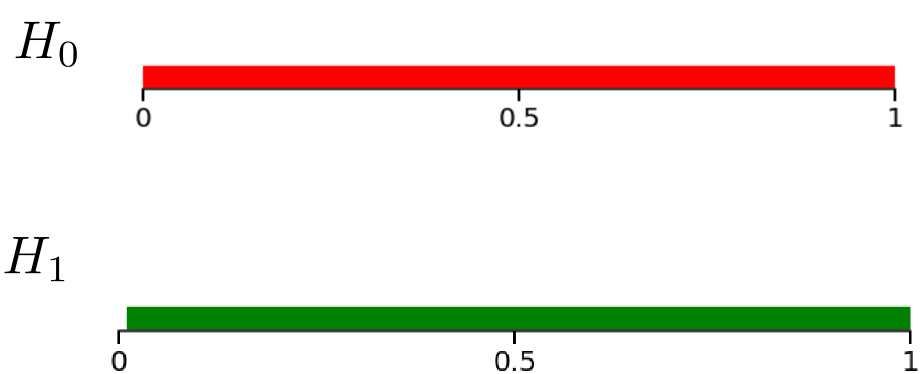
Last update: February 9, 2021

Introduction

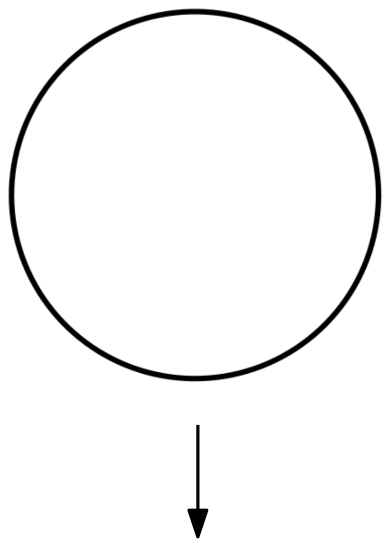
Let $X \subset \mathbb{R}^n$ finite, seen as a sample of $\mathcal{M}$.



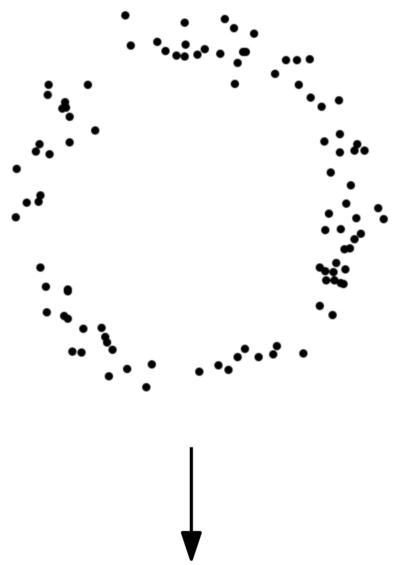
Barcodes of the Čech filtration

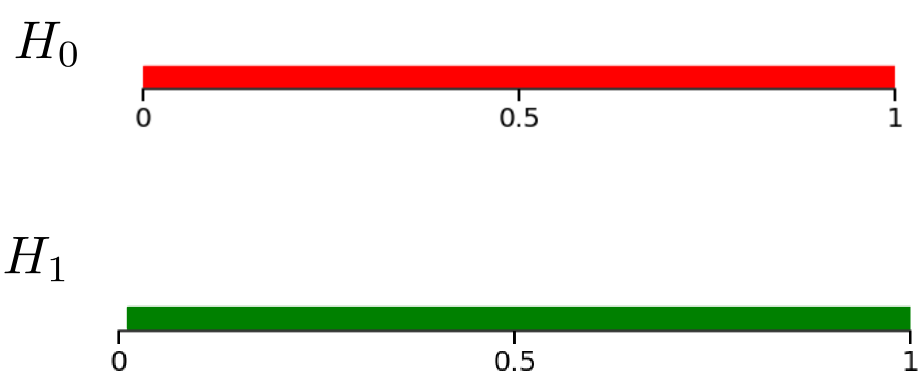


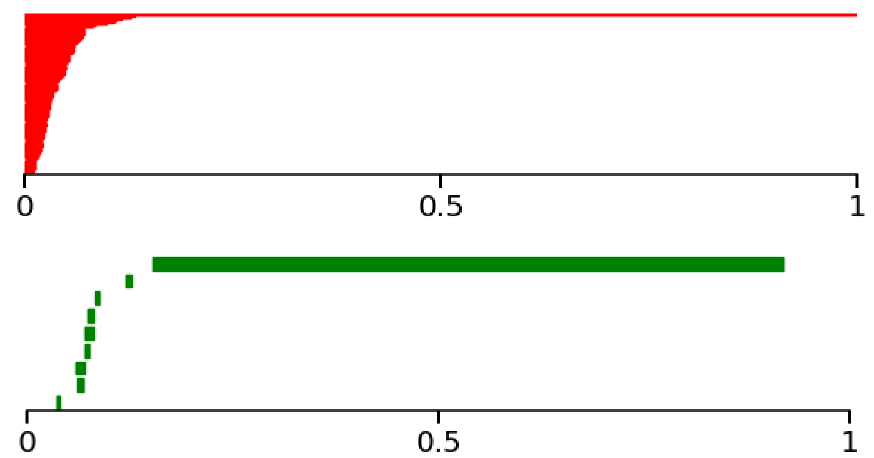
Let $X \subset \mathbb{R}^n$ finite, seen as a sample of $\mathcal{M}$.



Barcodes of the Čech filtration







stability

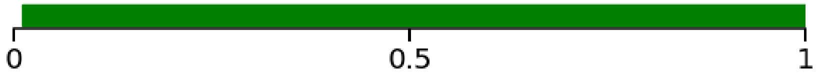
I - Distances between persistence modules

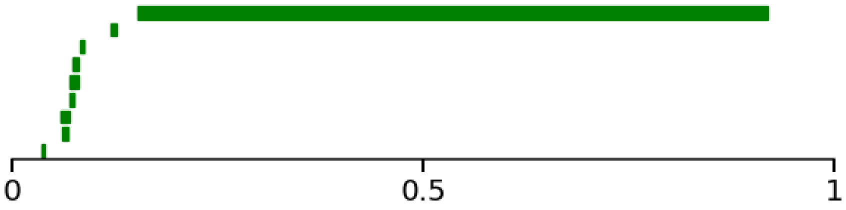
II - Isometry theorem

III - Stability theorem

Bottleneck distance

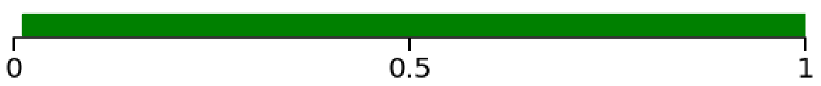
Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.

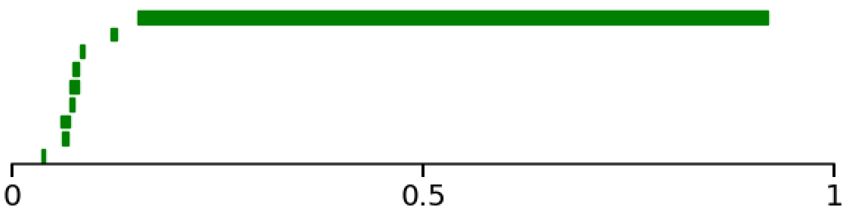




Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.





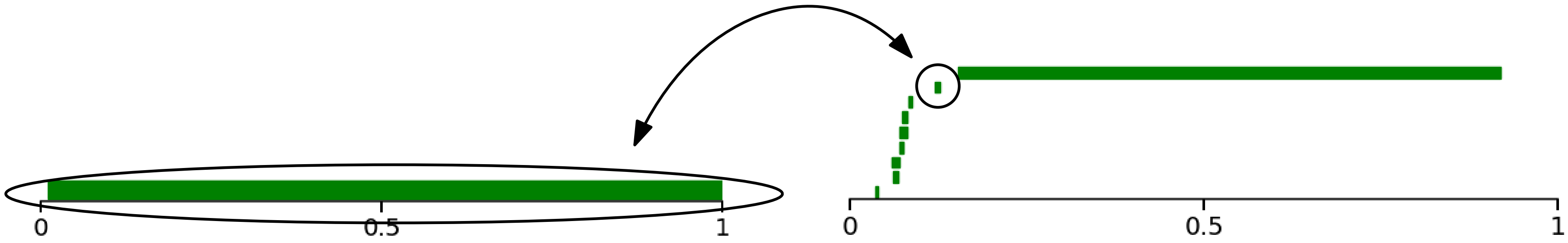
A **partial matching** between the barcodes is a subset $M \subset P \times Q$ such that

- for every $p \in P$, there exists at most one $q \in Q$ such that $(p, q) \in M$,
- for every $q \in Q$, there exists at most one $p \in P$ such that $(p, q) \in M$.

The points $p \in P$ (resp. $q \in Q$) such that there exists $q \in Q$ (resp. $p \in P$) with $(p, q) \in M$ are said **matched** by M.

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.



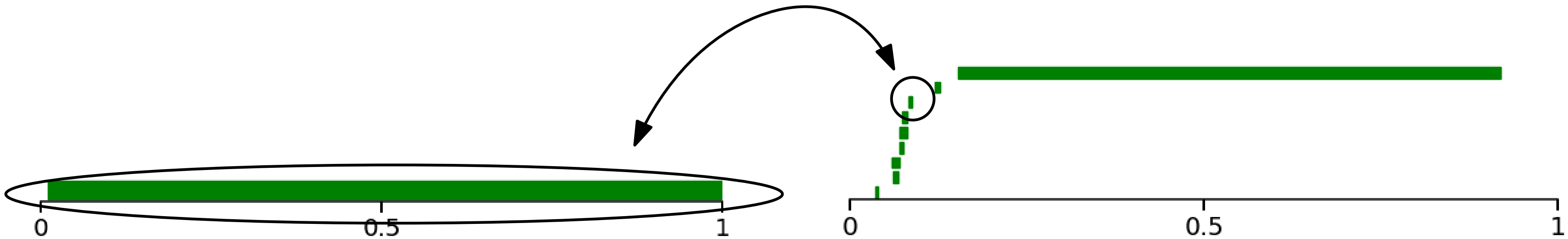
A **partial matching** between the barcodes is a subset $M \subset P \times Q$ such that

- for every $p \in P$, there exists at most one $q \in Q$ such that $(p, q) \in M$,
- for every $q \in Q$, there exists at most one $p \in P$ such that $(p, q) \in M$.

The points $p \in P$ (resp. $q \in Q$) such that there exists $q \in Q$ (resp. $p \in P$) with $(p, q) \in M$ are said **matched** by M.

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.



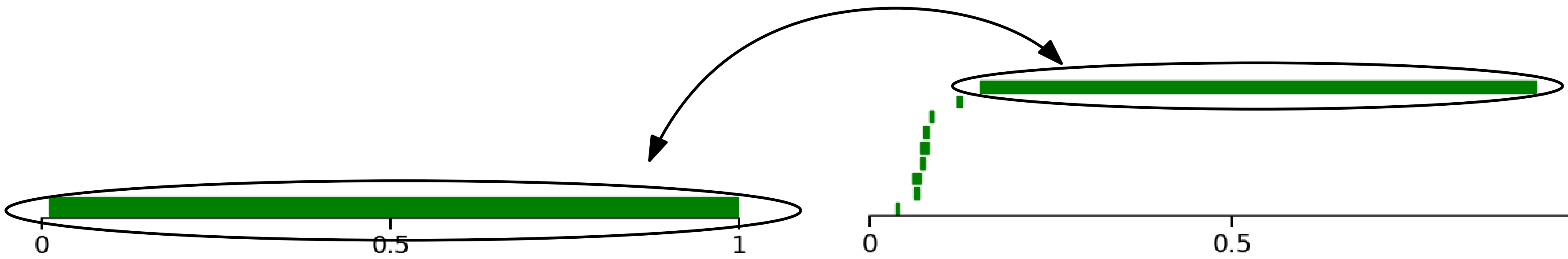
A **partial matching** between the barcodes is a subset $M \subset P \times Q$ such that

- for every $p \in P$, there exists at most one $q \in Q$ such that $(p, q) \in M$,
- for every $q \in Q$, there exists at most one $p \in P$ such that $(p, q) \in M$.

The points $p \in P$ (resp. $q \in Q$) such that there exists $q \in Q$ (resp. $p \in P$) with $(p, q) \in M$ are said **matched** by M.

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.



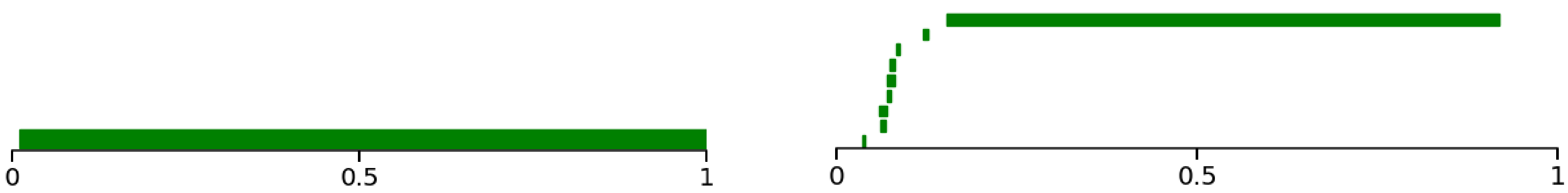
A **partial matching** between the barcodes is a subset $M \subset P \times Q$ such that

- for every $p \in P$, there exists at most one $q \in Q$ such that $(p, q) \in M$,
- for every $q \in Q$, there exists at most one $p \in P$ such that $(p, q) \in M$.

The points $p \in P$ (resp. $q \in Q$) such that there exists $q \in Q$ (resp. $p \in P$) with $(p, q) \in M$ are said **matched** by M.

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.



A **partial matching** between the barcodes is a subset $M \subset P \times Q$ such that

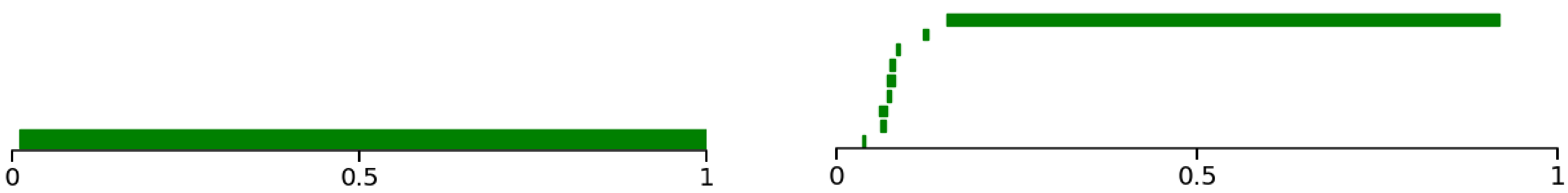
- for every $p \in P$, there exists at most one $q \in Q$ such that $(p, q) \in M$,
- for every $q \in Q$, there exists at most one $p \in P$ such that $(p, q) \in M$.

The points $p \in P$ (resp. $q \in Q$) such that there exists $q \in Q$ (resp. $p \in P$) with $(p, q) \in M$ are said **matched** by M.

If a point $p \in P$ (resp. $q \in Q$) is not matched by M, we consider that it is matched with the singleton $\overline{p} = \left[\frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}\right]$ (resp. $\overline{q} = \left[\frac{q_1+q_2}{2}, \frac{q_1+q_2}{2}\right]$).

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.



A **partial matching** between the barcodes is a subset $M \subset P \times Q$ such that

- for every $p \in P$, there exists at most one $q \in Q$ such that $(p, q) \in M$,
- for every $q \in Q$, there exists at most one $p \in P$ such that $(p, q) \in M$.

The points $p \in P$ (resp. $q \in Q$) such that there exists $q \in Q$ (resp. $p \in P$) with $(p, q) \in M$ are said **matched** by M.

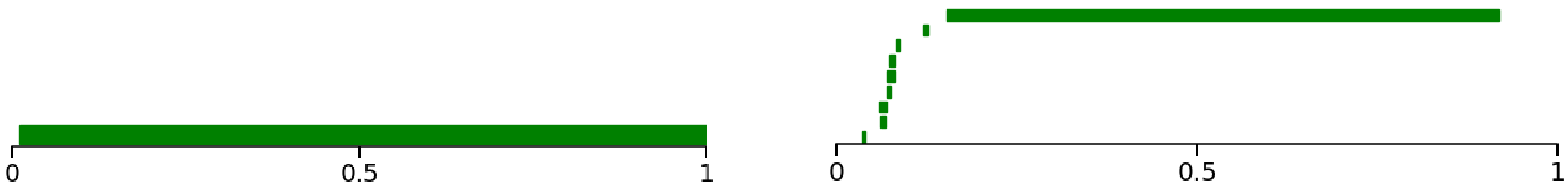
If a point $p \in P$ (resp. $q \in Q$) is not matched by M, we consider that it is matched with the singleton $\overline{p} = \left[\frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}\right]$ (resp. $\overline{q} = \left[\frac{q_1+q_2}{2}, \frac{q_1+q_2}{2}\right]$).

The **cost** of a matched pair (p, q) (resp. $(p, \overline{p})$, resp. $(\overline{q}, q)$) is the sup norm $\|p - q\|_\infty = \sup\{|p_1 - q_1|, |p_2 - q_2|\}$ (resp. $\|p - \overline{p}\|_\infty$, resp. $\|\overline{q} - q\|_\infty$).

The **cost** of the partial matching M, denoted $\mathrm{cost}(M)$, is the supremum of all costs.

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.

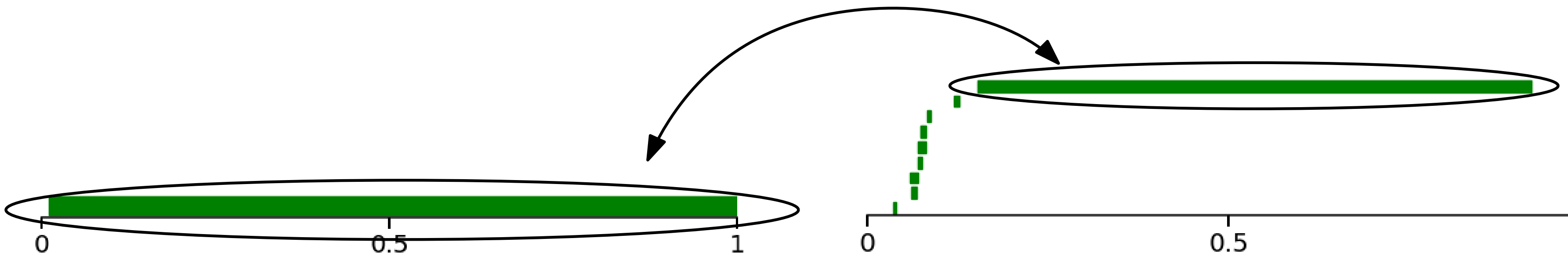


Definition: The *bottleck distance* between P and Q is defined as the infimum of costs over all the partial matchings:

$$\mathrm{d_b}\,(P, Q) = \inf\{\mathrm{cost}(M),\ M \text{ is a partial matching between } P \text{ and } Q\}.$$

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.

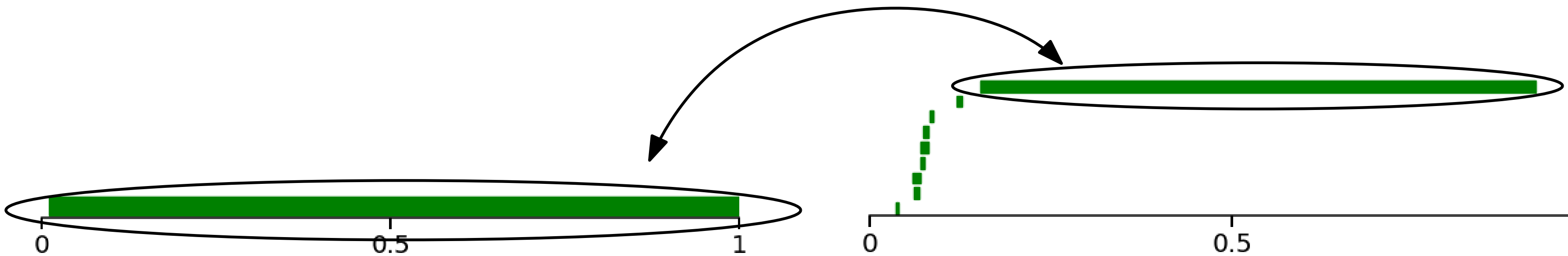


Definition: The *bottleck distance* between P and Q is defined as the infimum of costs over all the partial matchings:

$$\mathrm{d_b}\,(P, Q) = \inf\{\mathrm{cost}(M),\ M \text{ is a partial matching between } P \text{ and } Q\}.$$

Bottleneck distance

Consider two barcodes P and Q, that is, multisets of intervals $\{(a_i, b_i), i \in \mathcal{I}\}$ of $(\overline{\mathbb{R}^+})^2$ such that $a_i \leq b_i$ for all $i \in \mathcal{I}$.



Definition: The *bottleck distance* between P and Q is defined as the infimum of costs over all the partial matchings:

$$\mathrm{d_b}\left(P, Q\right) = \inf\{\mathrm{cost}(M),\ M \text{ is a partial matching between } P \text{ and } Q\}.$$

If $\mathbb{U}$ and $\mathbb{V}$ are two decomposable persistence modules, we define their *bottleneck distance* as

$$\mathrm{d_b}\left(\mathbb{U}, \mathbb{V}\right) = \mathrm{d_b}\left(\mathrm{Diagram}\left(\mathbb{U}\right), \mathrm{Diagram}\left(\mathbb{V}\right)\right).$$

Bottleneck distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Define the barcodes $P = \{[a,b]\}$ and $Q = \{[a',b']\}$.

First, consider the empty matching $M = \emptyset$. The intervals are matched to their midpoint, and the cost is

$$\left|(a,b) - \left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right|_\infty = \frac{b-a}{2}, \qquad \left|(a',b') - \left(\frac{a'+b'}{2}, \frac{a'+b'}{2}\right)\right|_\infty = \frac{b'-a'}{2}$$

The total cost is $\operatorname{cost}(M) = \max\left\{\frac{b-a}{2}, \frac{b'-a'}{2}\right\}$.

Next, consider the matching $M' = \left\{\big((a,b),(a',b')\big)\right\}$. The intervals are matched together, and the cost of the pair is

$$|(a,b) - (a',b')|_\infty = \max\{|a-a'|, |b-b'|\}.$$

which is also $\operatorname{cost}(M')$.

These are the only two partial matchings, and we deduce the bottleneck distance

$$\mathrm{d_b}(P,Q) = \min\left\{\max\left\{\frac{b-a}{2}, \frac{b'-a'}{2}\right\}, \max\{|a-a'|, |b-b'|\}\right\}.$$

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Define the barcodes $P = \{[a, b]\}$ and $Q = \{[a', b']\}$. We have

$$\mathrm{d_b}\left(P, Q\right) = \min\left\{\max\left\{\frac{b-a}{2}, \frac{b'-a'}{2}\right\}, \max\{|a-a'|, |b-b'|\}\right\}.$$

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a, b]$ and $\mathbb{B}[a', b']$.

Their barcodes are the sets P and Q of the previous example, from which we deduce

$$\mathrm{d_b}\left(\mathbb{B}[a, b], \mathbb{B}[a', b']\right) = \min\left\{\max\left\{\frac{b-a}{2}, \frac{b'-a'}{2}\right\}, \max\{|a-a'|, |b-b'|\}\right\}.$$

Interleaving distance

Consider two persistence modules $\mathbb{V}$ and $\mathbb{W}$:

$$\dashrightarrow V^{t_1} \xrightarrow{v_{t_1}^{t_2}} V^{t_2} \xrightarrow{v_{t_2}^{t_3}} V^{t_3} \xrightarrow{v_{t_3}^{t_4}} V^{t_4} \;\text{-----}$$

$$\dashrightarrow W^{t_1} \xrightarrow{w_{t_1}^{t_2}} W^{t_2} \xrightarrow{w_{t_2}^{t_3}} W^{t_3} \xrightarrow{w_{t_3}^{t_4}} W^{t_4} \;\text{-----}$$

Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t \colon V^t \to W^{t+\epsilon})_{t \in \mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$\begin{tikzcd}
V^s \arrow[r, "v_s^t"] \arrow[d, "\phi_s"] & V^t \arrow[d, "\phi_t"] \\
W^{s+\epsilon} \arrow[r, "w_{s+\epsilon}^{t+\epsilon}"] & W^{t+\epsilon}
\end{tikzcd}$$

Interleaving distance

Consider two persistence modules $\mathbb{V}$ and $\mathbb{W}$:

$$\cdots\cdots \dashrightarrow V^{t_1} \xrightarrow{v_{t_1}^{t_2}} V^{t_2} \xrightarrow{v_{t_2}^{t_3}} V^{t_3} \xrightarrow{v_{t_3}^{t_4}} V^{t_4} \cdots\cdots$$

$$\cdots\cdots \dashrightarrow W^{t_1} \xrightarrow{w_{t_1}^{t_2}} W^{t_2} \xrightarrow{w_{t_2}^{t_3}} W^{t_3} \xrightarrow{w_{t_3}^{t_4}} W^{t_4} \cdots\cdots$$

Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t \colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$\begin{tikzcd} V^s \arrow[r, "v_s^t"] \arrow[d, "\phi_s"] & V^t \arrow[d, "\phi_t"] \\ W^{s+\epsilon} \arrow[r, "w_{s+\epsilon}^{t+\epsilon}"] & W^{t+\epsilon} \end{tikzcd}$$

An ϵ-**interleaving** between $\mathbb{V}$ and $\mathbb{W}$ is a pair of ϵ-morphisms $(\phi_t \colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ and $(\psi_t \colon W^t \to V^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagrams commute for every $t \in \mathbb{R}^+$:

$$\begin{tikzcd} V^t \arrow[rr, "v_t^{t+2\epsilon}"] \arrow[dr, "\phi_t"] & & V^{t+2\epsilon} \\ & W^{t+\epsilon} \arrow[ur, "\psi_{t+\epsilon}"] & \end{tikzcd} \qquad \begin{tikzcd} & V^{t+\epsilon} \arrow[dr, "\phi_{t+\epsilon}"] & \\ W^t \arrow[ur, "\psi_t"] \arrow[rr, "w_t^{t+2\epsilon}"] & & W^{t+2\epsilon} \end{tikzcd}$$

Consider two persistence modules $\mathbb{V}$ and $\mathbb{W}$:

$$\cdots\cdots \dashrightarrow V^{t_1} \xrightarrow{v_{t_1}^{t_2}} V^{t_2} \xrightarrow{v_{t_2}^{t_3}} V^{t_3} \xrightarrow{v_{t_3}^{t_4}} V^{t_4} \cdots\cdots$$

$$\cdots\cdots \dashrightarrow W^{t_1} \xrightarrow{w_{t_1}^{t_2}} W^{t_2} \xrightarrow{w_{t_2}^{t_3}} W^{t_3} \xrightarrow{w_{t_3}^{t_4}} W^{t_4} \cdots\cdots$$

Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t \colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$\begin{tikzcd}
V^s \arrow[r, "v_s^t"] \arrow[d, "\phi_s"] & V^t \arrow[d, "\phi_t"] \\
W^{s+\epsilon} \arrow[r, "w_{s+\epsilon}^{t+\epsilon}"] & W^{t+\epsilon}
\end{tikzcd}$$

An ϵ-**interleaving** between $\mathbb{V}$ and $\mathbb{W}$ is a pair of ϵ-morphisms $(\phi_t \colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ and $(\psi_t \colon W^t \to V^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagrams commute for every $t \in \mathbb{R}^+$:

$$\begin{tikzcd}
V^t \arrow[rr, "v_t^{t+2\epsilon}"] \arrow[dr, "\phi_t"'] & & V^{t+2\epsilon} \\
& W^{t+\epsilon} \arrow[ur, "\psi_{t+\epsilon}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^{t+\epsilon} \arrow[dr, "\phi_{t+\epsilon}"] & \\
W^t \arrow[ur, "\psi_t"] \arrow[rr, "w_t^{t+2\epsilon}"] & & W^{t+2\epsilon}
\end{tikzcd}$$

The **interleaving distance** is: $\mathrm{d_i}(\mathbb{V}, \mathbb{W}) = \inf\{\epsilon \geq 0, \mathbb{V} \text{ and } \mathbb{W} \text{ are } \epsilon\text{-interleaved}\}$.

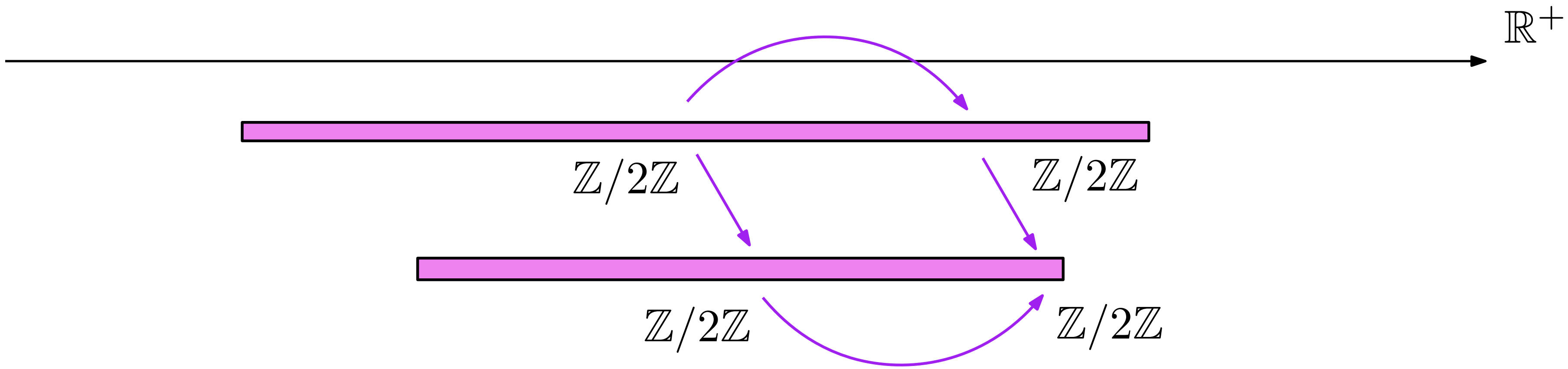
Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a, b]$ and $\mathbb{B}[a', b']$.

Let us find an ϵ-interleaving.

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a, b]$ and $\mathbb{B}[a', b']$.

Let us find an ϵ-interleaving.



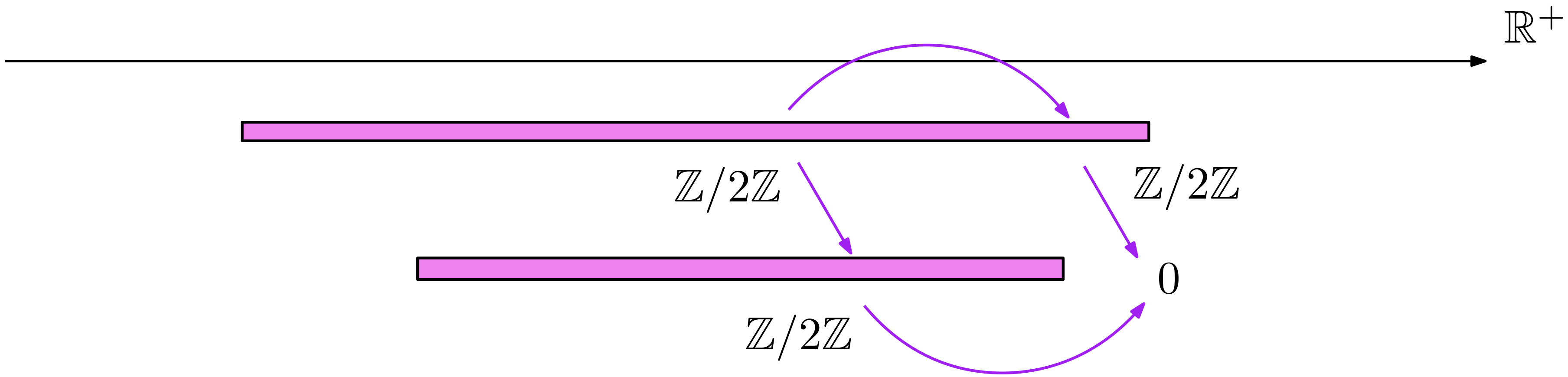
Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t \colon V^t \to W^{t+\epsilon})_{t \in \mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$\begin{tikzcd}
V^s \arrow[r, "v_s^t"] \arrow[d, "\phi_s"] & V^t \arrow[d, "\phi_t"] \\
W^{s+\epsilon} \arrow[r, "w_{s+\epsilon}^{t+\epsilon}"] & W^{t+\epsilon}
\end{tikzcd}$$

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a,b]$ and $\mathbb{B}[a',b']$.

Let us find an ϵ-interleaving.



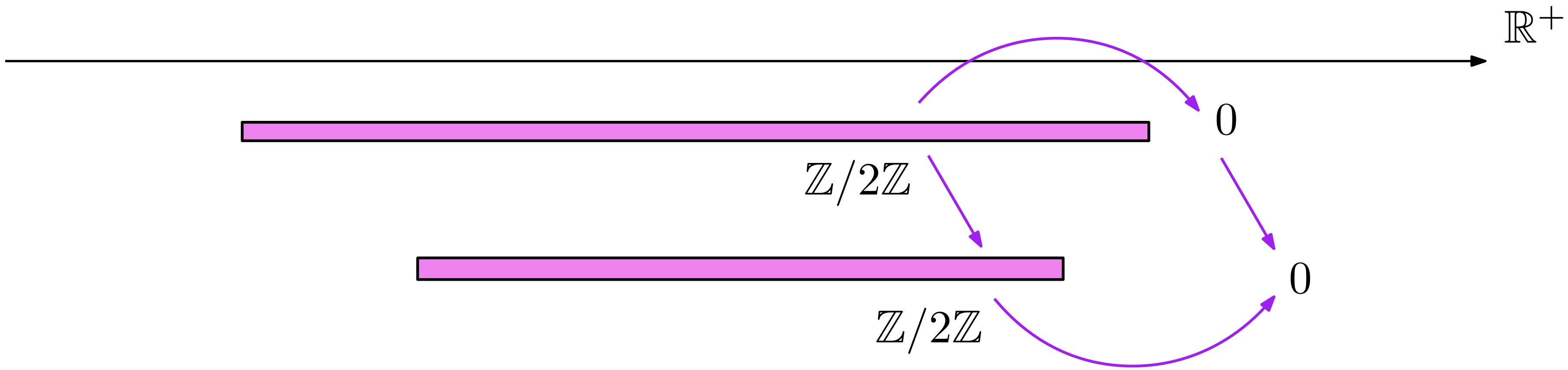
Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t \colon V^t \to W^{t+\epsilon})_{t \in \mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$\begin{tikzcd} V^s \arrow[r, "v_s^t"] \arrow[d, "\phi_s"] & V^t \arrow[d, "\phi_t"] \\ W^{s+\epsilon} \arrow[r, "w_{s+\epsilon}^{t+\epsilon}"] & W^{t+\epsilon} \end{tikzcd}$$

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a, b]$ and $\mathbb{B}[a', b']$.

Let us find an ϵ-interleaving.



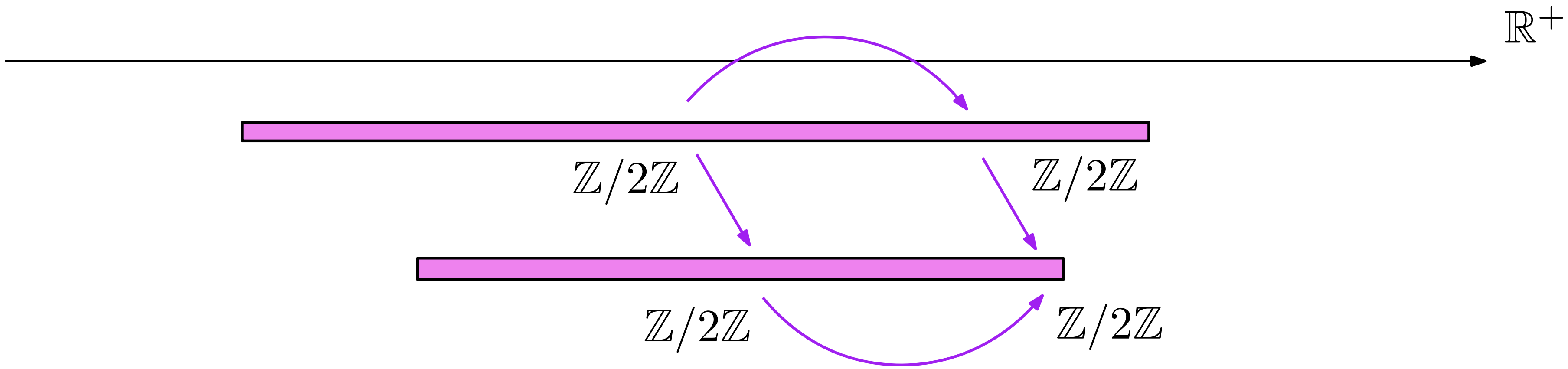
Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t \colon V^t \to W^{t+\epsilon})_{t \in \mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

$$\begin{tikzcd} V^s \arrow[r, "v_s^t"] \arrow[d, "\phi_s"] & V^t \arrow[d, "\phi_t"] \\ W^{s+\epsilon} \arrow[r, "w_{s+\epsilon}^{t+\epsilon}"] & W^{t+\epsilon} \end{tikzcd}$$

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a, b]$ and $\mathbb{B}[a', b']$.

Let us find an ϵ-interleaving.



Given $\epsilon \geq 0$, an ϵ-**morphism** between $\mathbb{V}$ and $\mathbb{W}$ is a family of linear maps $\phi = (\phi_t\colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagram commutes for every $s \leq t \in \mathbb{R}^+$:

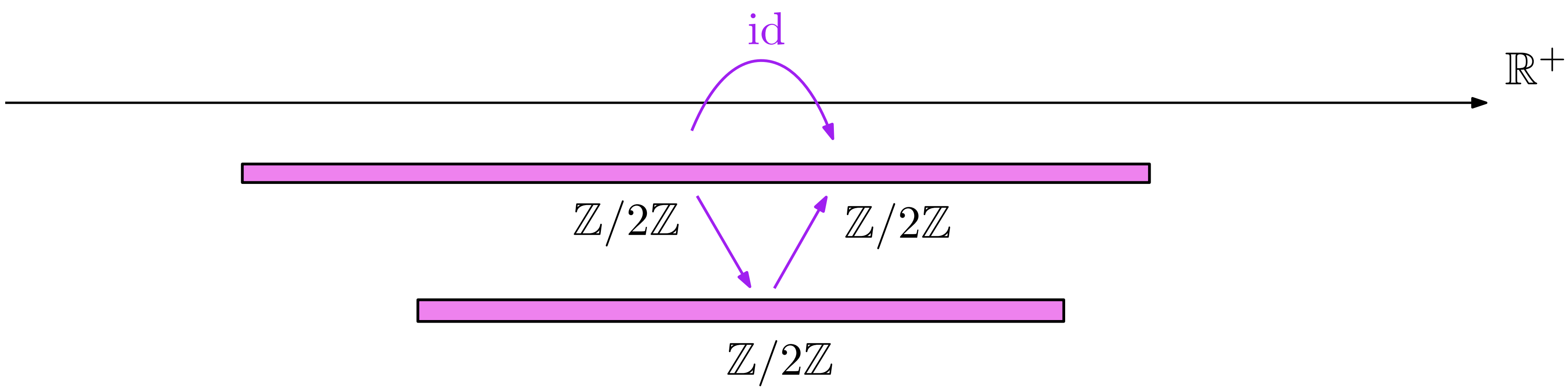
$$\begin{array}{ccc} V^s & \xrightarrow{v_s^t} & V^t \\ \downarrow \phi_s & & \downarrow \phi_t \\ W^{s+\epsilon} & \xrightarrow{w_{s+\epsilon}^{t+\epsilon}} & W^{t+\epsilon} \end{array}$$

⟶ Only two possibilities for ϕ:

- always the zero map
- always nonzero when V^t and $W^{t+\epsilon}$ are nonzero

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a,b]$ and $\mathbb{B}[a',b']$.

Let us find an ϵ-interleaving.



An ϵ-**interleaving** between $\mathbb{V}$ and $\mathbb{W}$ is a pair of ϵ-morphisms $(\phi_t \colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ and $(\psi_t \colon W^t \to V^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagrams commute for every $t \in \mathbb{R}^+$:

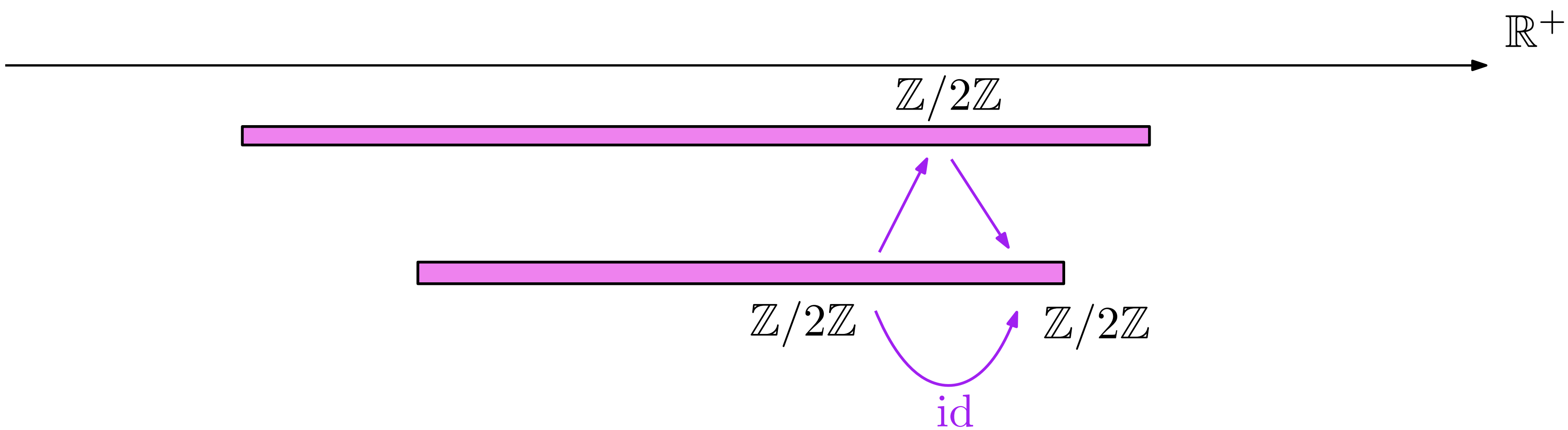
$$\begin{tikzcd}
V^t \arrow[rr, "v_t^{t+2\epsilon}"] \arrow[dr, "\phi_t"'] & & V^{t+2\epsilon} \\
& W^{t+\epsilon} \arrow[ur, "\psi_{t+\epsilon}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^{t+\epsilon} \arrow[dr, "\phi_{t+\epsilon}"] & \\
W^t \arrow[ur, "\psi_t"] \arrow[rr, "w_t^{t+2\epsilon}"] & & W^{t+2\epsilon}
\end{tikzcd}$$

$\longrightarrow$ $\psi_{t+\epsilon} \circ \phi_t$ must be nonzero when $[t, t+\epsilon] \subset [a,b]$

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a,b]$ and $\mathbb{B}[a',b']$.

Let us find an ϵ-interleaving.



An ϵ-**interleaving** between $\mathbb{V}$ and $\mathbb{W}$ is a pair of ϵ-morphisms $(\phi_t \colon V^t \to W^{t+\epsilon})_{t\in\mathbb{R}^+}$ and $(\psi_t \colon W^t \to V^{t+\epsilon})_{t\in\mathbb{R}^+}$ such that the following diagrams commute for every $t \in \mathbb{R}^+$:

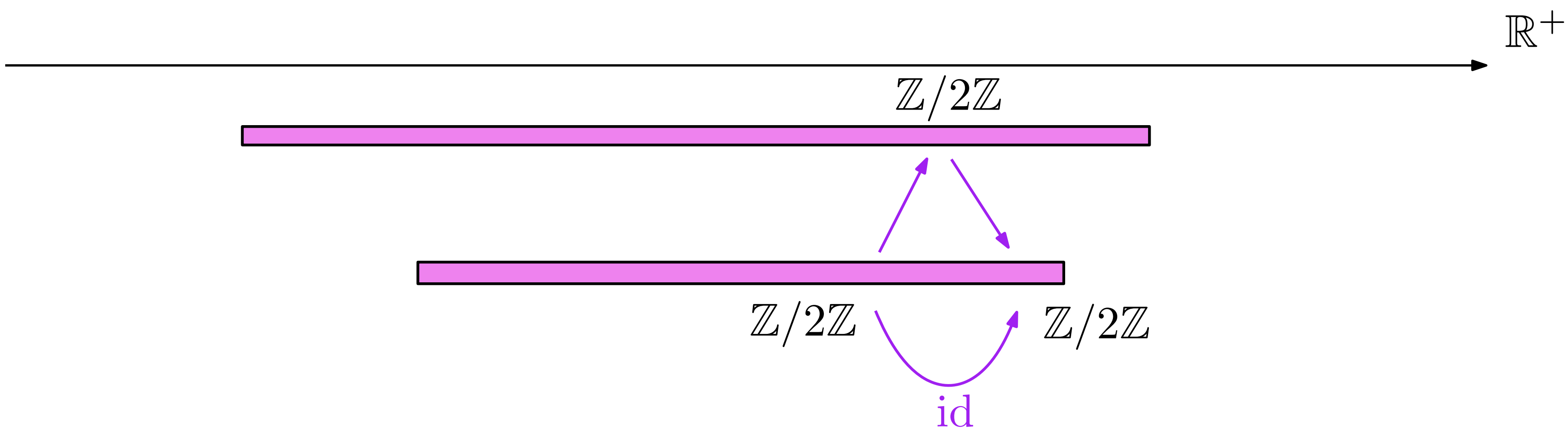
$$\begin{tikzcd}
V^t \arrow[rr, "v_t^{t+2\epsilon}"] \arrow[dr, "\phi_t"'] & & V^{t+2\epsilon} \\
& W^{t+\epsilon} \arrow[ur, "\psi_{t+\epsilon}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^{t+\epsilon} \arrow[dr, "\phi_{t+\epsilon}"] & \\
W^t \arrow[ur, "\psi_t"] \arrow[rr, "w_t^{t+2\epsilon}"] & & W^{t+2\epsilon}
\end{tikzcd}$$

$\longrightarrow$ $\psi_{t+\epsilon} \circ \phi_t$ must be nonzero when $[t, t+\epsilon] \subset [a,b]$

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a,b]$ and $\mathbb{B}[a',b']$.

Let us find an ϵ-interleaving.



An ϵ-**interleaving** between $\mathbb{V}$ and $\mathbb{W}$ is a pair of ϵ-morphisms $(\phi_t \colon V^t \to W^{t+\epsilon})_{t \in \mathbb{R}^+}$ and $(\psi_t \colon W^t \to V^{t+\epsilon})_{t \in \mathbb{R}^+}$ such that the following diagrams commute for every $t \in \mathbb{R}^+$:

$$\begin{tikzcd}
V^t \arrow[rr, "v_t^{t+2\epsilon}"] \arrow[dr, "\phi_t"'] & & V^{t+2\epsilon} \\
& W^{t+\epsilon} \arrow[ur, "\psi_{t+\epsilon}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V^{t+\epsilon} \arrow[dr, "\phi_{t+\epsilon}"] & \\
W^t \arrow[ur, "\psi_t"] \arrow[rr, "w_t^{t+2\epsilon}"] & & W^{t+2\epsilon}
\end{tikzcd}$$

$\psi_{t+\epsilon} \circ \phi_t$ must be nonzero when $[t, t+\epsilon] \subset [a,b]$

$\phi_{t+\epsilon} \circ \psi_t$ must be nonzero when $[t, t+\epsilon] \subset [a',b']$

Interleaving distance

Example: Let $a, a', b, b' \in \mathbb{R}^+$ such that $a \leq b$ and $a' \leq b'$. Consider the interval-modules $\mathbb{B}[a,b]$ and $\mathbb{B}[a',b']$.

Let us find an ϵ-interleaving.

$\longrightarrow$ Only two possibilities for ϕ:
- always the zero map
- always nonzero when V^t and $W^{t+\epsilon}$ are nonzero

$\longrightarrow$ $\psi_{t+\epsilon} \circ \phi_t$ must be nonzero when $[t, t+\epsilon] \subset [a,b]$
$\phi_{t+\epsilon} \circ \psi_t$ must be nonzero when $[t, t+\epsilon] \subset [a',b']$

We deduce that either
- $|a-b| \leq 2\epsilon$ and $|a'-b'| \leq 2\epsilon$, or
- $|a-a'| \leq \epsilon$ and $|b-b'| \leq \epsilon$

Conclusion: $\mathrm{d_i}\left(\mathbb{B}[a,b], \mathbb{B}[a',b']\right) = \min\left\{\max\left\{\frac{b-a}{2}, \frac{b'-a'}{2}\right\}, \max\{|a-a'|, |b-b'|\}\right\}$

I - Distances between persistence modules

II - Isometry theorem

III - Stability theorem

Isometry theorem

Theorem (Chazal, de Silva, Glisse, Oudot, 2009): If the persistence modules $\mathbb{U}$ and $\mathbb{V}$ are interval-decomposable, then $\mathrm{d_i}\left(\mathbb{U},\mathbb{V}\right) = \mathrm{d_b}\left(\mathbb{U},\mathbb{V}\right)$.

→ *Stability*: $\mathrm{d_i}\left(\mathbb{U},\mathbb{V}\right) \geq \mathrm{d_b}\left(\mathbb{U},\mathbb{V}\right)$

→ *Converse stability*: $\mathrm{d_i}\left(\mathbb{U},\mathbb{V}\right) \leq \mathrm{d_b}\left(\mathbb{U},\mathbb{V}\right)$

Isometry theorem

Theorem (Chazal, de Silva, Glisse, Oudot, 2009): If the persistence modules $\mathbb{U}$ and $\mathbb{V}$ are interval-decomposable, then $\mathrm{d_i}(\mathbb{U},\mathbb{V}) = \mathrm{d_b}(\mathbb{U},\mathbb{V})$.

Proof: Let us write the decomposition of the persistence modules in intervals:

$$\mathbb{V} \simeq \bigoplus_{I\in\mathcal{I}} \mathbb{B}[I] \qquad\qquad \mathbb{W} \simeq \bigoplus_{J\in\mathcal{J}} \mathbb{B}[J]$$

Suppose that we have a ϵ-partial matching $M \subset \mathcal{I}\times\mathcal{J}$. This gives a matching of some intervals (I,J), where $I=(a,b)$ and $J=(a',b')$, such that $|a-a'|\leq\epsilon$ and $|b-b'|\leq\epsilon$.

We can build an ϵ-interleaving between $\mathbb{B}[I]$ and $\mathbb{B}[J]$, that we denote $(\phi_{(I,J)},\psi_{(I,J)})$.

Some intervals I (resp. J) are not matched, in which case their length is not greater than 2ϵ, and we can build an ϵ-interleaving with the zero persistence module. We denote this interleaving $(\phi_{(I,0)},\psi_{(I,0)})$ (resp. $(\phi_{(0,J)},\psi_{(0,J)})$).

Now, let us consider the sums of all these linear maps:

$$\overline{\phi} = \bigoplus_{(I,J)\text{ matched}} \phi_{(I,J)} \bigoplus_{I\text{ not matched}} \phi_{(I,0)}, \qquad \overline{\psi} = \bigoplus_{(I,J)\text{ matched}} \psi_{(I,J)} \bigoplus_{J\text{ not matched}} \phi_{(0,J)}$$

$\longrightarrow$ $(\overline{\phi},\overline{\psi})$ is an ϵ-interleaving $\longrightarrow$ $\mathrm{d_i}(\mathbb{U},\mathbb{V}) \leq \mathrm{d_b}(\mathbb{U},\mathbb{V})$

Isometry theorem

Theorem (Chazal, de Silva, Glisse, Oudot, 2009): If the persistence modules $\mathbb{U}$ and $\mathbb{V}$ are interval-decomposable, then $\mathrm{d_i}(\mathbb{U}, \mathbb{V}) = \mathrm{d_b}(\mathbb{U}, \mathbb{V})$.

$\longrightarrow$ *Stability*: $\mathrm{d_i}(\mathbb{U}, \mathbb{V}) \geq \mathrm{d_b}(\mathbb{U}, \mathbb{V})$

$\longrightarrow$ *Converse stability*: $\mathrm{d_i}(\mathbb{U}, \mathbb{V}) \leq \mathrm{d_b}(\mathbb{U}, \mathbb{V})$

The stability part is more difficult.

A first strategy uses the interpolation lemma, and concludes with the box lemma.

Interpolation lemma: If $\mathbb{U}$ and $\mathbb{V}$ are δ-interleaved, then there exists a family of persistence modules $(\mathbb{U}_t)_{t\in[0,\delta]}$ such that $\mathbb{U}_0 = \mathbb{U}$, $\mathbb{U}_\delta = \mathbb{V}$ and $\mathrm{d_i}(\mathbb{U}_s, \mathbb{U}_t) \leq |s-t|$ for every $s, t \in [0, \delta]$.

Another proof builds an explicit partial matching from an interleaving (Bauer, Lesnick, 2013).

I - Distances between persistence modules

II - Isometry theorem

III - Stability theorem

Back to the thickenings

Let X and Y be two subets of $\mathbb{R}^n$. Define $\epsilon = \mathrm{d_H}\left(X, Y\right)$ (Hausdorff distance).

We have seen that $X \subset Y^\epsilon$ and $Y \subset X^\epsilon$. We even have that $X^t \subset Y^{t+\epsilon}$ and $Y^t \subset X^{t+\epsilon}$ for all $t \geq 0$.

By denoting j and k these inclusions, we have a commutative diagram

$$
\begin{tikzcd}
{} \arrow[r, dashed] & X^t \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & X^{t+2\epsilon} \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & X^{t+4\epsilon} \arrow[rr, hook, dashed] \arrow[dr, hook, "j_{t+4\epsilon}"'] & & {} \\
{} \arrow[rr, dashed] & & Y^{t+\epsilon} \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & Y^{t+3\epsilon} \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & Y^{t+5\epsilon} \arrow[r, hook] & {}
\end{tikzcd}
$$

Back to the thickenings

Let X and Y be two subets of $\mathbb{R}^n$. Define $\epsilon = \mathrm{d_H}(X, Y)$ (Hausdorff distance).

We have seen that $X \subset Y^\epsilon$ and $Y \subset X^\epsilon$. We even have that $X^t \subset Y^{t+\epsilon}$ and $Y^t \subset X^{t+\epsilon}$ for all $t \geq 0$.

By denoting j and k these inclusions, we have a commutative diagram

$$
\begin{tikzcd}
{} \arrow[r, dashed] & X^t \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & X^{t+2\epsilon} \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & X^{t+4\epsilon} \arrow[dr, hook, "j_{t+4\epsilon}"'] \arrow[rr, hook, dashed] & & {} \\
{} \arrow[rr, dashed] & & Y^{t+\epsilon} \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & Y^{t+3\epsilon} \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & Y^{t+5\epsilon} \arrow[r, hook, dashed] & {}
\end{tikzcd}
$$

This also gives inclusions between Čech complexes:

$$
\begin{tikzcd}
\check{\mathrm{C}}\mathrm{ech}^t(X) \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X) \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X) \arrow[dr, hook, "j_{t+4\epsilon}"'] \arrow[r, hook, dashed] & {} \\
{} \arrow[r, dashed] & \check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y)
\end{tikzcd}
$$

Back to the thickenings

[...] This also gives inclusions between Čech complexes:

$$
\begin{tikzcd}
\check{\mathrm{C}}\mathrm{ech}^{t}(X) \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X) \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X) \arrow[r, hook, dashed] \arrow[dr, hook, "j_{t+4\epsilon}"'] & {} \\
{} \arrow[r, dashed] & \check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y)
\end{tikzcd}
$$

Now, we apply the i^{th} homology functor.

$$
\begin{tikzcd}
H_i(\check{\mathrm{C}}\mathrm{ech}^{t}(X)) \arrow[rr] \arrow[dr, "(j_t)_*"'] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X)) \arrow[rr] \arrow[dr, "(j_{t+2\epsilon})_*"'] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X)) \arrow[r, dash, dashed] \arrow[dr, "(j_{t+4\epsilon})_*"'] & {} \\
{} \arrow[r, dashed] & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+\epsilon})_*"] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+3\epsilon})_*"] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y))
\end{tikzcd}
$$

Back to the thickenings

[...] This also gives inclusions between Čech complexes:

$$
\begin{tikzcd}[column sep=small]
\check{\mathrm{C}}\mathrm{ech}^{t}(X) \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X) \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X) \arrow[r, hook, dashed] \arrow[dr, hook, "j_{t+4\epsilon}"'] & {} \\
{} \arrow[r, dashed] & \check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y)
\end{tikzcd}
$$

Now, we apply the i^{th} homology functor.

$$
\begin{tikzcd}[column sep=small]
H_i(\check{\mathrm{C}}\mathrm{ech}^{t}(X)) \arrow[rr] \arrow[dr, "(j_t)_*"'] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X)) \arrow[rr] \arrow[dr, "(j_{t+2\epsilon})_*"'] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X)) \arrow[r, dashed, no head] \arrow[dr, "(j_{t+4\epsilon})_*"'] & {} \\
{} \arrow[r, dashed] & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+\epsilon})_*"] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+3\epsilon})_*"] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y))
\end{tikzcd}
$$

persistence module of Čech complex of X

[...] This also gives inclusions between Čech complexes:

$$\begin{tikzcd}
& \text{Čech}^{t}(X) \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & \text{Čech}^{t+2\epsilon}(X) \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & \text{Čech}^{t+4\epsilon}(X) \arrow[r, hook, dashed, no head] \arrow[dr, hook, "j_{t+4\epsilon}"'] & {} \\
{} \arrow[rr, dashed] & & \text{Čech}^{t+\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & \text{Čech}^{t+3\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & \text{Čech}^{t+5\epsilon}(Y)
\end{tikzcd}$$

Now, we apply the i^{th} homology functor.

$$\begin{tikzcd}
& H_i(\text{Čech}^{t}(X)) \arrow[rr] \arrow[dr, "(j_t)_*"'] & & H_i(\text{Čech}^{t+2\epsilon}(X)) \arrow[rr] \arrow[dr, "(j_{t+2\epsilon})_*"'] & & H_i(\text{Čech}^{t+4\epsilon}(X)) \arrow[r, dashed, no head] \arrow[dr, "(j_{t+4\epsilon})_*"'] & {} \\
{} \arrow[rr, dashed] & & H_i(\text{Čech}^{t+\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+\epsilon})_*"] & & H_i(\text{Čech}^{t+3\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+3\epsilon})_*"] & & H_i(\text{Čech}^{t+5\epsilon}(Y))
\end{tikzcd}$$

persistence module of Čech complex of X

persistence module of Čech complex of Y

Back to the thickenings

[...] This also gives inclusions between Čech complexes:

$$\begin{tikzcd}
\check{\mathrm{C}}\mathrm{ech}^{t}(X) \arrow[rr, hook] \arrow[dr, hook, "j_t"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X) \arrow[rr, hook] \arrow[dr, hook, "j_{t+2\epsilon}"'] & & \check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X) \arrow[r, hook, dashed, no head] \arrow[dr, hook, "j_{t+4\epsilon}"'] & {} \\
{} \arrow[r, dashed] & \check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y) \arrow[rr, hook] \arrow[ur, hook, "k_{t+3\epsilon}"] & & \check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y)
\end{tikzcd}$$

Now, we apply the i^{th} homology functor.

$$\begin{tikzcd}
H_i(\check{\mathrm{C}}\mathrm{ech}^{t}(X)) \arrow[rr] \arrow[dr, "(j_t)_*"'] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+2\epsilon}(X)) \arrow[rr] \arrow[dr, "(j_{t+2\epsilon})_*"'] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+4\epsilon}(X)) \arrow[r, dashed, no head] \arrow[dr, "(j_{t+4\epsilon})_*"'] & {} \\
{} \arrow[r, dashed] & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+\epsilon})_*"] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+3\epsilon}(Y)) \arrow[rr] \arrow[ur, "(k_{t+3\epsilon})_*"] & & H_i(\check{\mathrm{C}}\mathrm{ech}^{t+5\epsilon}(Y))
\end{tikzcd}$$

persistence module of Čech complex of X

persistence module of Čech complex of Y

ϵ-interleaving between the persistence modules

Back to the thickenings

[...]

Hence the persistence modules $\underbrace{\left(H_i(\check{\mathrm{C}}\mathrm{ech}^t(X))\right)_{t\geq 0}}_{\mathbb{U}}$ and $\underbrace{\left(H_i(\check{\mathrm{C}}\mathrm{ech}^t(Y))\right)_{t\geq 0}}_{\mathbb{V}}$ are ϵ-interleaved.

Hence $\mathrm{d_i}\left(\mathbb{U}, \mathbb{V}\right) \leq \epsilon$.

Back to the thickenings

[...]

Hence the persistence modules $\Big(H_i(\check{\mathrm{C}}\mathrm{ech}^t(X))\Big)_{t\geq 0}$ and $\Big(H_i(\check{\mathrm{C}}\mathrm{ech}^t(Y))\Big)_{t\geq 0}$ are ϵ-interleaved.

$\mathbb{U}$ $\mathbb{V}$

Hence $\mathrm{d_i}\left(\mathbb{U}, \mathbb{V}\right) \leq \epsilon$.

We use the isometry theorem: $\mathrm{d_b}\left(\mathbb{U}, \mathbb{V}\right) = \mathrm{d_i}\left(\mathbb{U}, \mathbb{V}\right) \leq \epsilon$.

Back to the thickenings

[...]

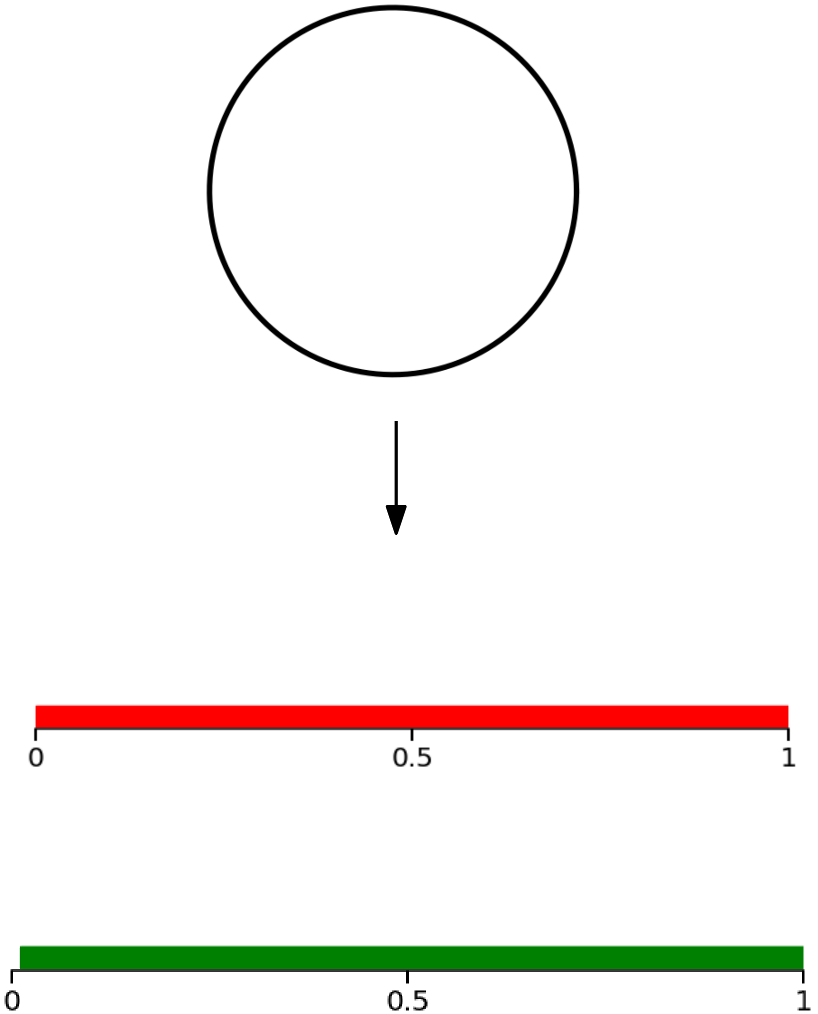
Hence the persistence modules $\underbrace{\left(H_i(\check{\mathrm{C}}\mathrm{ech}^t(X))\right)_{t\geq 0}}_{\mathbb{U}}$ and $\underbrace{\left(H_i(\check{\mathrm{C}}\mathrm{ech}^t(Y))\right)_{t\geq 0}}_{\mathbb{V}}$ are ϵ-interleaved.

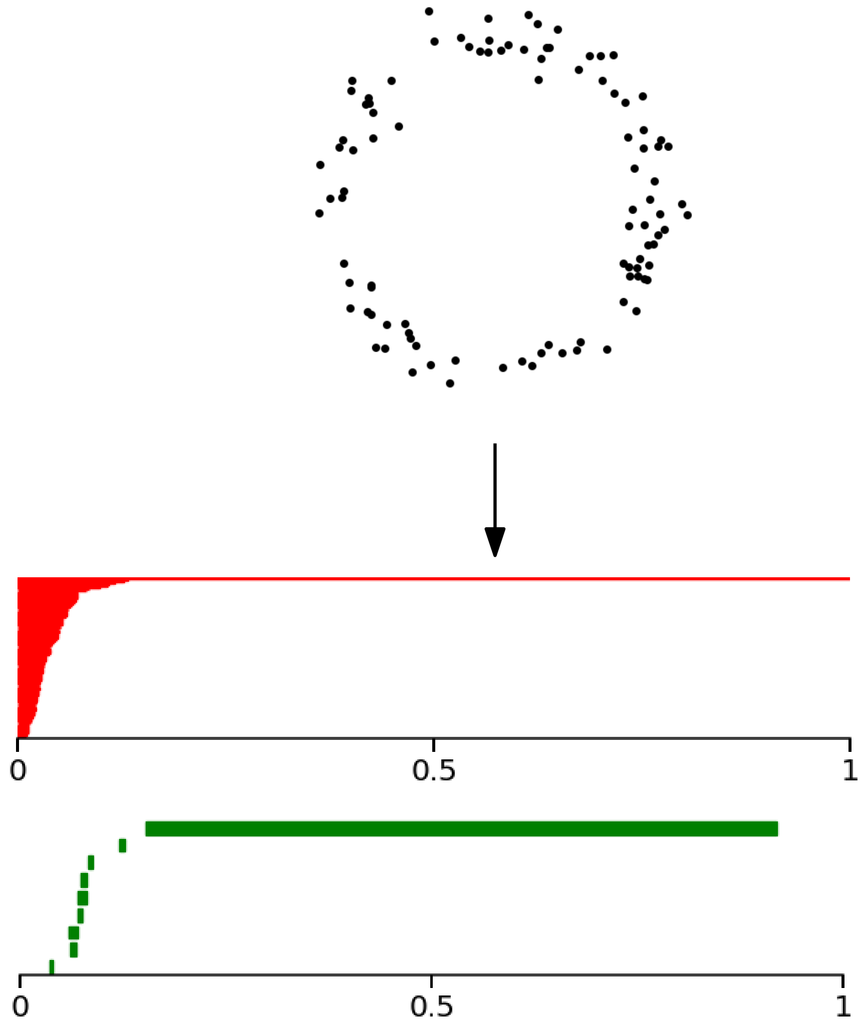
Hence $\mathrm{d_i}\left(\mathbb{U}, \mathbb{V}\right) \leq \epsilon$.

We use the isometry theorem: $\mathrm{d_b}\left(\mathbb{U}, \mathbb{V}\right) = \mathrm{d_i}\left(\mathbb{U}, \mathbb{V}\right) \leq \epsilon$.

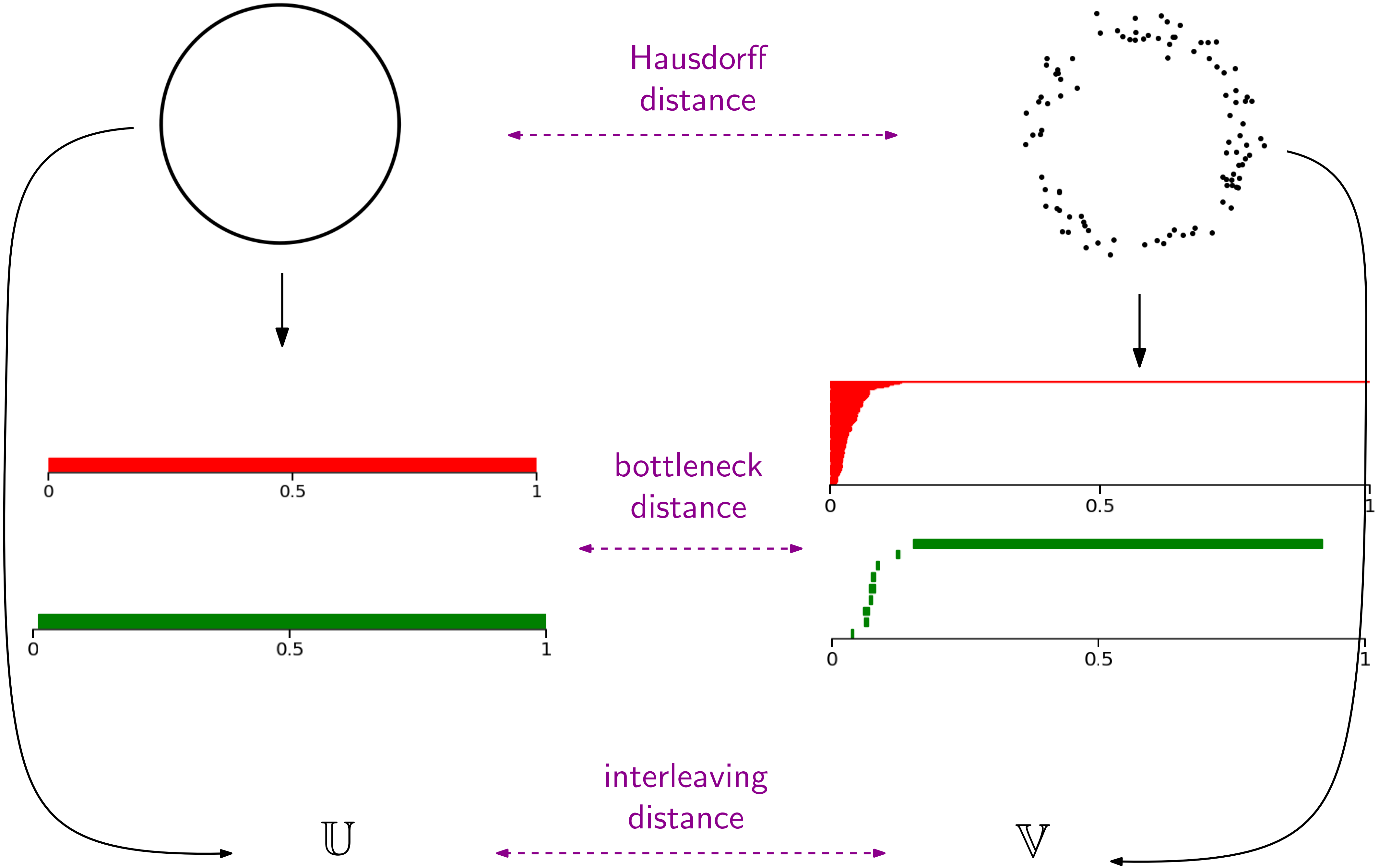
Theorem (Cohen-Steiner, Edelsbrunner, Harer, 2005): Let X and Y be two subsets of $\mathbb{R}^n$. Consider their Čech (resp. Rips) filtrations, and the corresponding i^{th} homology persistence modules, $\mathbb{U}$ and $\mathbb{V}$. Suppose that they are interval-decomposables. Then $\mathrm{d_b}\left(\mathbb{U}, \mathbb{V}\right) \leq \mathrm{d_H}\left(X, Y\right)$.

Summary





Summary


Hausdorff
distance
0
0.5
1
0
0.5
1
bottleneck
distance
0
0.5
1
0
0.5
1
interleaving
distance
$\mathbb{U}$
$\mathbb{V}$

Conclusion

We interpreted topological noise as small bars in barcodes.

We defined a distance between barcodes that is not too sensitive to small bars.

We linked this distance with an algebraic-flavoured distance.

We deduced a satisfactory result of stability.

Homework: Exercise 53

Conclusion

We interpreted topological noise as small bars in barcodes.

We defined a distance between barcodes that is not too sensitive to small bars.

We linked this distance with an algebraic-flavoured distance.

We deduced a satisfactory result of stability.

Homework: Exercise 53

Last lesson tomorrow!

Merci !