Homework

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Exercise 8. We want to show that $B(0,1) \subset \mathbb{R}^n$ and $\mathbb{R}^n$ are homeomorphic. For this, we will show that

$$f : \mathbb{R}^n \to B(0,1)$$
$$x \mapsto \frac{\|x\|}{(\|x\|+1)^2} x$$

is a homeomorphism between the two spaces. Initially, we will prove that, for each $a \in B(0,1)$, there is exactly one $b \in \mathbb{R}^n$ such that $f(b) = a$, and thus conclude that f is indeed a bijection (it is injective, since, by its definition, $f(x) = f(y) \iff x = y$). Then, we will find f^{-1} and show that it is continuous (the function itself is continuous, as we can see by its definition). So, we want to solve $a = \frac{\|b\|b}{(\|b\|+1)^2}$ for b; some algebraic manipulations show that $b = \frac{a}{\sqrt{\|a\|}(1-\sqrt{\|a\|})}$ (since $\|a\| = \left(\frac{\|b\|}{1+\|b\|}\right)^2 < 1$, the denominator is positive). Therefore, for each $a \in B(0,1)$, there is a (unique) $b \in \mathbb{R}^n$ such that $f(b) = a$; thus, f is a bijection. Moreover, we found that

$$f^{-1}(x) = \frac{x}{\sqrt{\|x\|}(1-\sqrt{\|x\|})},$$

a continuous function in $B(0,1)$. Since f is a continuous bijection with continuous inverse, it is a homeomorphism; this proves that $B(0,1)$ and $\mathbb{R}^n$ are homeomorphic.

Exercise 11. We want to show that $[0,1)$ and $(0,1)$ are not homeomorphic. For contradiction, suppose that there is a homeomorphism $f : [0,1) \to (0,1)$. Let, then,

$$g : (0,1) \to (0,1) \setminus f(0).$$

Since $g = f|_{(0,1)}$, g is a homeomorphism. However, $(0,1)$ has one connected component, and $(0,1) \setminus f(0)$ has two (since $(0,1) \setminus f(0) = (0, f(0)) \cup (f(0), 1)$ is the union of two disjoint connected open sets); absurd, because homeomorphic sets have the same number of connected components. Therefore, $[0,1)$ and $(0,1)$ aren't homeomorphic.