Homework 1

Topological Data Analysis with Persistent Homology

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Exercise 4. Consider a point $z \in \mathcal{B}\left(\frac{x+y}{2}, \frac{r}{2}\right)$, that is, $\left\| z - \frac{x+y}{2} \right\| < \frac{r}{2}$. We know that $\left\| x - \frac{x+y}{2} \right\| = \left\| \frac{x-y}{2} \right\| = \frac{r}{2}$. This way, we see that

$$\begin{aligned} \|z - x\| &= \left\| z - \frac{x+y}{2} + \frac{x+y}{2} - x \right\| \\ &\leq \left\| z - \frac{x+y}{2} \right\| + \left\| \frac{x+y}{2} - x \right\| \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

Therefore, $z \in \mathcal{B}(x, r)$.

Similarly, we can show that $z \in \mathcal{B}(y, r)$. So we conclude that $z \in \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$, which means $\mathcal{B}\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$.

Exercise 5. Consider an open ball $\mathcal{B}(x, r)$ of $\mathbb{R}^n$. Take an arbitrary point inside it, let's call it y. So $\|x - y\| < r$.

Now, consider the open ball $\mathcal{B}(y, d)$, with $d = r - \|x - y\|$. Because $\|x - y\| < r$, we have $d > 0$. If we take a point z inside this open ball, we will have $\|y - z\| < d = r - \|x - y\|$. Well, it's the same of $\|y - z\| + \|x - y\| < r$, that is, by triangle inequality, $\|x - z\| < r$. We can say that $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, d) \subset \mathcal{B}(x, r)$.

We can conclude that, for any point y of the open ball $\mathcal{B}(x, r)$, there is another open ball $\mathcal{B}(y, d)$ that contains y but also $\mathcal{B}(y, d) \subset \mathcal{B}(x, r)$, that is, $\mathcal{B}(x, r)$ is open.